A Novel Chaotic Synchronization Scheme Based on Impulsive Stability Theory

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Abstract—**This paper introduces a new chaotic synchronization scheme based on impulsive stability theory. The synchronization of two identical chaotic systems with parametric uncertainties and mismatch is considered. Some sufficient conditions are developed to guarantee the robust synchronization of the chaotic systems. Compared with existing schemes, the amount of information transmitted between the two systems is reduced significantly. A numerical example is presented to demonstrate the effectiveness of the method.**

Index Terms—**chaotic, synchronization, impulsive stability, uncertainties**

I. INTRODUCTION

Chaos is very interesting nonlinear phenomenon. Due to its potential technological applications, synchronization of chaotic systems has been studied with increasing interest in the past few years, in variety areas of bioengineering, computer science, social science, economy and so on.

Since Pecora and Carroll[1] introduced a method to synchronize two identical chaotic systems with different initial conditions, various synchronization schemes have been studied by many researchers, such as OGY method [2], observer-based control[3], adaptive control[4], feedback control[5], backstepping control[6,7], and so on. All of these methods have adopted the continuous chaotic synchronization scheme. To increase the efficiency of bandwidth usage, impulsive chaotic synchronization [8,9] has been proposed. In synchronization process, the control signals are transmitted from driving system to driven system only at discrete time instants. Thus reduces the amount of information transmitted between the two systems. Various theoretical and experimental results of impulsive chaotic synchronization can be found in[8-14]. Further studies on how to reduce the synchronization information are implemented by researchers. Basically in three ways, enlarge the fixed impulse intervals[15], time-varying impulse intervals to enlarge the average impulse interval[16,17], impulsive synchronization via partial chaotic states[18].

In this paper, we propose a new synchronization scheme based on impulsive stability theory. Due to the parametric mismatch and uncertainties, the magnitude of the synchronization error between two chaotic systems will oscillate between certain bound. If the synchronization error can be kept within the pre-specified synchronization threshold, the two chaotic systems could be considered synchronized. The process of synchronization can be divided into two stages. In the first stage, the magnitude of the synchronization error approach to the pre-specified bound from the initial value. In the second stage, the magnitude of the synchronization error stay within the pre-specified bound. We found that in the second stage the full chaotic states of the driving system is not necessarily to be transmitted to the driven system, as the error between two systems is bounded and only one bit information can indicate such error. Thus in our scheme, only one bit information is needed to transmit from the transmitter to the receiver for each synchronization impulse. As a result, the amount of information transmitted between the two systems is reduced significantly.

This paper is organized as follows. In section II, we formulate the problem and propose the overall system model. The condition which provides quantitative relation between the synchronization threshold, the impulse interval, the bounds of parametric uncertainties and mismatch and impulse synchronization coefficient, is derived in section III and section IV. In section V, a numerical example is given, and conclusions are given in section VI.

II. SYSTEM MODEL

In this section, the model of chaotic synchronization scheme that is essentially based on impulsive stability theory is presented. The block diagram of the proposed system is shown in Fig.1.

The proposed chaotic synchronization system is composed of two main parts: the transmitter and the

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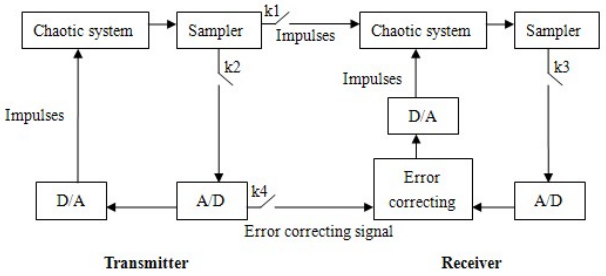

Figure 1. Block diagram of the chaotic synchronization scheme.

receiver. The transmitter is composed of a chaotic system, a sampler, an A/D module and a D/A module. The receiver has a chaotic system, a sampler, an A/D module, a D/A module and an error correcting module. The system is running as follows: Step 1, Turn on k1, chaotic states of the transmitter are sampled and transmitted to the receiver as synchronization pulses. Turn off k1 after the chaotic system in the receiver is synchronized with the chaotic system in the transmitter. Step 2, turn on k2, k3 and k4, chaotic states of the transmitter are sampled and then quantized by the A/D module. The error correcting signals which gained from the digitized chaotic states are then sent to the receiver. The D/A module convert the digitized chaotic states to synchronization pulses. In the receiver side, chaotic states are sampled and then quantized by the A/D module. The digitized chaotic states which modified by the error correcting signals are then converted to synchronization pulses by the D/A module.

The two chaotic systems in the transmitter and the receiver are implemented with the Chen chaotic system in this paper. However, the method can also be used while the chaotic systems in the transmitter and the receiver are other chaotic systems.

The state equation of the chaotic system with parametric uncertainties in the transmitter side is given by

$$\begin{cases} \dot{\mathbf{X}}_1 = (\mathbf{A} + \Delta\mathbf{A}_1(t))\mathbf{X}_1 + \varphi(\mathbf{X}_1) & t \neq \tau_i \\ \mathbf{X}_1(\tau_i) = \mathbf{X}_1(\tau_i^-) \quad i = 1,2,\ldots,p \\ \mathbf{X}_1(\tau_i) = \mathbf{X}_1(\tau_i^-) - \mathbf{B}(\mathbf{X}_1(\tau_i^-) - Q(\mathbf{X}_1(\tau_i^-))) \quad i = p+1, p+2,\ldots \end{cases} \tag{1}$$

Where $\mathbf{X}_1 \in \boldsymbol{R}^3$ is the state vector and $\mathrm{A} \in \boldsymbol{R}^3$ is a nominal constant matrix, and $\varphi(\cdot)$ is the nonlinear part of Chen system

$$\mathbf{X}_1 = [x_1, y_1, z_1]^T,\ \mathbf{A} = \begin{bmatrix} -a & a & 0 \\ c-a & c & 0 \\ 0 & 0 & -b \end{bmatrix},\ \varphi(\mathbf{X}_1) = \begin{bmatrix} 0 \\ -x_1 z_1 \\ x_1 y_1 \end{bmatrix}$$

$\Delta\mathbf{A}_1(t)$ denotes time-varying parametric uncertainties in the transmitter side, where:

$$\Delta\mathbf{A}_1 = \begin{bmatrix} -\Delta a_1(t) & \Delta a_1(t) & 0 \\ \Delta c_1(t) - \Delta a_1(t) & \Delta c_1(t) & 0 \\ 0 & 0 & -\Delta b_1(t) \end{bmatrix}$$

The impulse synchronization coefficient **B** denotes the pulse intensity, is a 3×3 diagonal matrix

$$\mathbf{B} = \begin{bmatrix} b_1 & & \\ & b_2 & \\ & & b_3 \end{bmatrix} \qquad b_1 \in (0,1),\ b_2 \in (0,1),\ b_3 \in (0,1)\ .$$

$Q(\cdot)$ is a predefined quantizer which converts a real number to a binary number with finite precision, $\{\tau_i\}(1 \le i \le \infty)$ satisfy $0<\tau_1<\tau_2<\ldots\tau_i<\tau_{i+1}<\ldots$, $\tau_i \to \infty$ as $i \to \infty$ with $\tau_i = \sum_{n=1}^{i} T_n$; T_n are impulsive time intervals; τ_i^- are the times immediately prior the times τ_i .

The state equation of the chaotic system with parametric uncertainties in the receiver side is given by

$$\begin{cases} \dot{\mathbf{X}}_2 = (\mathbf{A} + \Delta\mathbf{A}_2(t))\mathbf{X}_2 + \varphi(\mathbf{X}_2) & t \neq \tau_i \\ \mathbf{X}_2(\tau_i) = \mathbf{X}_2(\tau_i^-) - \mathbf{B}(\mathbf{X}_2(\tau_i^-) - \mathbf{X}_1(\tau_i^-)) \quad i = 1,2,\ldots,p \\ \mathbf{X}_2(\tau_i) = \mathbf{X}_2(\tau_i^-) - \mathbf{B}(\mathbf{X}_2(\tau_i^-) - C(\mathbf{X}_2(\tau_i^-), \mathbf{EC}(\tau_i^-))) \quad i = p+1, p+2,\ldots \end{cases} \tag{2}$$

Where $\Delta\mathbf{A}_2(t)$ denotes time-varying parametric uncertainties in the receiver side:

$$\Delta\mathbf{A}_2 = \begin{bmatrix} -\Delta a_2(t) & \Delta a_2(t) & 0 \\ \Delta c_2(t) - \Delta a_2(t) & \Delta c_2(t) & 0 \\ 0 & 0 & -\Delta b_2(t) \end{bmatrix}$$

$C(\cdot)$ is the error correcting function, and **EC** is the error correcting signal vector from the transmitter side.

Due to the parametric uncertainties of the chaotic systems, it is practical to set a pre-defined bound ε for the magnitude of synchronization error between the two chaotic systems. And we have some priori knowledge as given below.

Let **e** denotes the synchronization error vector, where:

$$\mathbf{e} = \begin{bmatrix} e_x(t) \\ e_y(t) \\ e_z(t) \end{bmatrix} = \begin{bmatrix} x_1(t) - x_2(t) \\ y_1(t) - y_2(t) \\ z_1(t) - z_2(t) \end{bmatrix}$$

Definition 1: The chaotic systems in transmitter side and receiver side are synchronized if $\exists\, t_0$, such that for $\forall t > t_0$

$$|e_x(t)| < \varepsilon, |e_y(t)| < \varepsilon, |e_z(t)| < \varepsilon$$

Assumption 1: The parametric uncertainties and mismatch $\Delta\mathbf{A}_1(t)$ and $\Delta\mathbf{A}_2(t)$ are bounded as follows:

$$|\Delta a_1(t)| \le \xi_a \cdot a, |\Delta a_2(t)| \le \xi_a \cdot a$$

$$|\Delta b_1(t)| \le \xi_b \cdot b, |\Delta b_2(t)| \le \xi_b \cdot b$$

$$|\Delta c_1(t)| \le \xi_c \cdot c, |\Delta c_2(t)| \le \xi_c \cdot c$$

where ξ is the error coefficient.

From (1) and (2), we can derive the error system

$$\begin{cases} \dot{\mathbf{e}} = ((\mathbf{A} + \Delta\mathbf{A}_1(t))\mathbf{X}_1 - (\mathbf{A} + \Delta\mathbf{A}_2(t))\mathbf{X}_2) + (\varphi(\mathbf{X}_1) - \varphi(\mathbf{X}_2)) & t \neq \tau_i \\ \mathbf{e}(\tau_i) = \mathbf{e}(\tau_i^-) - \mathbf{Be}(\tau_i^-) \quad i = 1,2,\ldots,p \\ \mathbf{e}(\tau_i) = \mathbf{e}(\tau_i^-) - \mathbf{B}(\mathbf{e}(\tau_i^-) - (C(\mathbf{X}_2(\tau_i^-), \mathbf{EC}(\tau_i^-)) - Q(\mathbf{X}_1(\tau_i^-))) \quad i = p+1, p+2,\ldots \end{cases} \tag{3}$$

In order to facilitate the analysis of stability of the error system (3), we re-write (3) as

$$\begin{cases} \dot{\mathbf{e}} = \mathbf{A}(t)\mathbf{e} + \Delta\mathbf{A}(t)\mathbf{X}_2 + \Psi(\mathbf{X}_1, \mathbf{X}_2) & t \neq \tau_i \\ \mathbf{e}(\tau_i) = \mathbf{e}(\tau_i^-) - \mathbf{Be}(\tau_i^-) \quad i = 1,2,\ldots,p \\ \mathbf{e}(\tau_i) = \mathbf{e}(\tau_i^-) - \mathbf{B}(\mathbf{e}(\tau_i^-) - (C(\mathbf{X}_2(\tau_i^-), \mathbf{EC}(\tau_i^-)) - Q(\mathbf{X}_1(\tau_i^-))) \quad i = p+1, p+2,\ldots \end{cases} \tag{4}$$

where:

$$\begin{cases} a(t)=a+\Delta a_1(t) \\ b(t)=b+\Delta b_1(t) \\ c(t)=c+\Delta c_1(t) \end{cases}, \ \mathbf{A}(t)=\begin{bmatrix} -a(t) & a(t) & 0 \\ c(t)-a(t) & c(t) & 0 \\ 0 & 0 & -b(t) \end{bmatrix},$$

$$\Psi(\mathbf{X}_1,\mathbf{X}_2)=\begin{pmatrix} 0 \\ x_1z_1-x_2z_2 \\ x_1y_1-x_2y_2 \end{pmatrix}, \ \begin{cases} \Delta a(t)=\Delta a_1(t)-\Delta a_2(t) \\ \Delta b(t)=\Delta b_1(t)-\Delta b_2(t) \\ \Delta c(t)=\Delta c_1(t)-\Delta c_2(t) \end{cases},$$

$$\Delta\mathbf{A}(t)=\begin{bmatrix} -\Delta a(t) & \Delta a(t) & 0 \\ \Delta c(t)-\Delta a(t) & \Delta c(t) & 0 \\ 0 & 0 & -\Delta b(t) \end{bmatrix}$$

According to Assumption 1, we have

$$|a(t)|\le(1+\xi_a)\cdot a,\ |\Delta a(t)|\le 2\xi_a\cdot a$$

$$|b(t)|\le(1+\xi_b)\cdot b,\ |\Delta b(t)|\le 2\xi_b\cdot b$$

$$|c(t)|\le(1+\xi_c)\cdot c,\ |\Delta c(t)|\le 2\xi_c\cdot c$$

III. ERROR CORRECTING SCHEME

To establish the synchronization of the chaotic systems, in this section, we proposed an error correcting scheme. The error correcting function C(·) is designed as follows.

$$C(\mathbf{X}_2(t),\mathbf{EC}(t))=\begin{bmatrix} Q(x_2(t)-\varepsilon) \\ Q(y_2(t)-\varepsilon) \\ Q(z_2(t)-\varepsilon) \end{bmatrix}+\begin{bmatrix} Q_l(x_1(t))\oplus Q_l(x_2(t)-\varepsilon) \\ Q_l(y_1(t))\oplus Q_l(y_2(t)-\varepsilon) \\ Q_l(z_1(t))\oplus Q_l(z_2(t)-\varepsilon) \end{bmatrix}Q(\delta)$$

where

$$\mathbf{EC}(t)=\begin{bmatrix} Q_l(x_1(t)) \\ Q_l(y_1(t)) \\ Q_l(z_1(t)) \end{bmatrix}$$

As $Q(\cdot)$ is a binary quantizer, $Q_l(x)$ represents the lowest bit of $Q(x)$. δ denotes the quantization step of the quantizer $Q(\cdot)$.

Lemma 1: If $|e_x(t)|<\varepsilon$, $|e_y(t)|<\varepsilon$, $|e_z(t)|<\varepsilon$, and $\delta\ge 2\varepsilon$, then $Q(\mathbf{X}_1(t))=C(\mathbf{X}_2(t),\mathbf{EC}(t))$

Proof: From the conditions of the lemma, we have

$$0<x_1(t)-(x_2(t)-\varepsilon)<2\varepsilon$$

$$0\le Q(x_1(t))-Q(x_2(t)-\varepsilon)\le Q(\delta)$$

Thus,

$$Q(x_1(t))=Q(x_2(t)-\varepsilon) \quad or \quad Q(x_1(t))=Q(x_2(t)-\varepsilon)+Q(\delta)$$

Case 1: $Q(x_1(t))=Q(x_2(t)-\varepsilon)$

Then

$$Q_l(x_1(t))=Q_l(x_2(t)-\varepsilon)$$

$$Q_l(x_1(t))\oplus Q_l(x_2(t)-\varepsilon)=0$$

Therefore

$$Q(x_1(t))=Q(x_2(t)-\varepsilon)+(Q_l(x_1(t))\oplus Q_l(x_2(t)-\varepsilon))Q(\delta)$$

Case 2: $Q(x_1(t))=Q(x_2(t)-\varepsilon)+Q(\delta)$

Then

$$Q_l(x_1(t))\ne Q_l(x_2(t)-\varepsilon)$$

$$Q_l(x_1(t))\oplus Q_l(x_2(t)-\varepsilon)=1$$

Therefore

$$Q(x_1(t))=Q(x_2(t)-\varepsilon)+(Q_l(x_1(t))\oplus Q_l(x_2(t)-\varepsilon))Q(\delta)$$

Similarly, we have

$$Q(y_1(t))=Q(y_2(t)-\varepsilon)+(Q_l(y_1(t))\oplus Q_l(y_2(t)-\varepsilon))Q(\delta)$$

$$Q(z_1(t))=Q(z_2(t)-\varepsilon)+(Q_l(z_1(t))\oplus Q_l(z_2(t)-\varepsilon))Q(\delta)$$

In summery, $Q(\mathbf{X}_1(t))=C(\mathbf{X}_2(t),\mathbf{EC}(t))$

IV. SYNCHRONIZATION SCHEME

In this section, we analyze the oscillation process of the error between the chaotic systems of the transmitter side and the receiver side. Furthermore a quantitative relationship between the length of the impulse intervals T_n and the synchronization coefficient b_1, b_2, b_3, synchronization threshold ε, the error coefficient ξ is established.

Lemma 2: If $\exists i_0\in \mathbf{Z}^+$, such that for $\forall i>i_0$

$$\|e(\tau_i^-)\| \triangleq |e_x(\tau_i^-)|+|e_y(\tau_i^-)|+|e_z(\tau_i^-)|<\varepsilon \tag{5}$$

then $\exists$ t_0, such that for $\forall t>t_0$

$$|e_x(t)|<\varepsilon, |e_y(t)|<\varepsilon, |e_z(t)|<\varepsilon$$

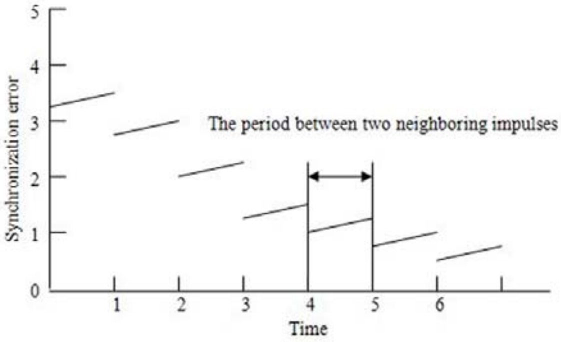

Figure 2. The synchronization error between two chaotic systems.

Proof: The process of the synchronization scheme can be divided into many periods which contain the impulse and the period between two neighboring impulses as shown in Fig.1.

Since the chaotic systems are sensitive to initial condition and parameters, the tiny error between two chaotic systems will increase with time. This phenomenon can be demonstrated by the schematic diagram shown in Fig.2. During the period between two neighboring impulses, the error between two chaotic systems is increased with time. Thus the error is largest at the time instants τ_i^-, such that for $\forall$ $i>i_0$

$$\|e(\tau_i^-)\|<\varepsilon$$

Therefore for $\forall t>\tau_{i0}^-$, $|e_x(t)|<\varepsilon$, $|e_y(t)|<\varepsilon$, $|e_z(t)|<\varepsilon$.

Lemma 3: If the following two conditions are satisfied

1) If $\|e(\tau_i^-)\|\ge\varepsilon$

$$\|e(\tau_{i+1}^-)\|<\|e(\tau_i^-)\| \tag{6}$$

2) If $\|e(\tau_i^-)\|<\varepsilon$

$$\|e(\tau_{i+1}^-)\|<\varepsilon \tag{7}$$

Then $\exists i_0\in \mathbf{Z}^+$, such that for $\forall t>\tau_{i0}^-$, $|e_x(t)|<\varepsilon$, $|e_y(t)|<\varepsilon$, $|e_z(t)|<\varepsilon$.

Proof: For $\forall |e_x(\tau_1)|$, $|e_y(\tau_1)|$, $|e_z(\tau_1)|$, to show the result, we consider two cases:

Case 1: $\|e(\tau_1)\|\ge\varepsilon$

Based on (6), the error between two chaotic systems decreases gradually till $\exists i_0 \in \mathbf{Z}^+$, such that

$$\left\| e(\tau_{i_0}^-) \right\| < \varepsilon$$

Once $\|e(\tau_{i0}^-)\| < \varepsilon$, from (7), for $\forall$ $i > i_0$, we have

$$\left\| e(\tau_i^-) \right\| < \varepsilon$$

According to **Lemma** 2, for $\forall t > \tau_{i0}^-$, $|e_x(t)| < \varepsilon$, $|e_y(t)| < \varepsilon$, $|e_z(t)| < \varepsilon$.

Case 2: $\|e(\tau_1)\| < \varepsilon$

Since $\|e(\tau_1)\| < \varepsilon$, from (7), for $\forall$ $i > 1$, we have

$$\left\| e(\tau_i^-) \right\| < \varepsilon$$

According to **Lemma** 2, for $\forall t > \tau_1$, $|e_x(t)| < \varepsilon$, $|e_y(t)| < \varepsilon$, $|e_z(t)| < \varepsilon$.

In summary, if condition 1 and 2 are satisfied, $\exists$ $i_0 \in \mathbf{Z}^+$, such that for $\forall$ $t > \tau_{i0}^-$, $|e_x(t)| < \varepsilon$, $|e_y(t)| < \varepsilon$, $|e_z(t)| < \varepsilon$.

Since the chaotic system we use is Chen system, it is obviously that $\sup(x_2(t)) > 0$, $\sup(y_2(t)) > 0$, $\sup(z_2(t)) > 0$, $\sup(e_x(t)) \geq 0$, $\sup(e_y(t)) \geq 0$, $\sup(e_z(t)) \geq 0$, and $\sup(x_2(t) - y_2(t)) > 0$, $\sup((c-a)x_2(t) - cy_2(t)) > 0$

Theorem 1: While the chaotic systems in both sides are Chen system, if the length of the impulse interval satisfied:

$$\Delta_1 < \frac{\min(b_1, b_2, b_3)}{\max(\mathrm{m}(\xi_c), \mathrm{n}(\xi_a, \xi_c), \mathrm{o}(\xi_b)) + C(\xi_a, \xi_b, \xi_c)/\varepsilon} \tag{8}$$

where:

$$\begin{aligned}
\mathrm{m}(\xi_c) &= c \cdot \xi_c + (c + \sup(y_2(t)) + \sup(z_2(t))) \\
\mathrm{n}(\xi_a, \xi_c) &= a \cdot \xi_a + c \cdot \xi_c + a + c + \sup(x_2(t)) + \sup(e_x(t)) \\
\mathrm{o}(\xi_b) &= b \cdot \xi_b + b + \sup(z_2(t)) + \sup(e_x(t)) \\
C(\xi_a, \xi_b, \xi_c) &= 2(a \cdot \xi_a + c \cdot \xi_c) \cdot \sup(x_2(t) - y_2(t)) + \\
&\quad 2a \cdot \xi_a \cdot \sup(x_2(t)) + 2b \cdot \xi_b \cdot \sup(z_2(t))
\end{aligned}$$

And the quantization step satisfied : $\delta \geq 2\varepsilon$

then $\exists$ t_0, such that for $\forall t > t_0$

$$|e_x(t)| < \varepsilon, |e_y(t)| < \varepsilon, |e_z(t)| < \varepsilon$$

Proof.

Case 1: $\|e(1 \cdot \Delta_1)\| \geq \varepsilon$

In this case, the error system between the transmitter and the receiver can be written as

$$\begin{cases} \dot{\mathbf{e}} = ((\mathbf{A} + \Delta\mathbf{A}_1(t))\mathbf{X}_1 - (\mathbf{A} + \Delta\mathbf{A}_2(t))\mathbf{X}_2) + (\varphi(\mathbf{X}_1) - \varphi(\mathbf{X}_2)) & t \neq \tau_i \\ \mathbf{e}(\tau_i) = \mathbf{e}(\tau_i^-) - \mathbf{B}\mathbf{e}(\tau_i^-) & i = 1, 2, \ldots, p \end{cases}$$

According to the result proposed in [15], we already know that the impulsive interval Δ_1 must be very short so that two chaotic systems can be synchronized with parametric uncertainties and mismatch. Thus, we have

$$\begin{aligned}
|e_x(t_{n+1})| &\approx (1 - b_1) \cdot |e_x(t_n)| + \Delta_1 \cdot |\dot{e}_x(t_n)| \\
|e_y(t_{n+1})| &\approx (1 - b_2) \cdot |e_y(t_n)| + \Delta_1 \cdot |\dot{e}_y(t_n)| \\
|e_z(t_{n+1})| &\approx (1 - b_3) \cdot |e_z(t_n)| + \Delta_1 \cdot |\dot{e}_z(t_n)|
\end{aligned} \tag{9}$$

From (9), we have

$$\begin{aligned}
|e_x(t_{n+1})| - |e_x(t_n)| &\approx \Delta_1 \cdot |\dot{e}_x(t_n)| - b_1 \cdot |e_x(t_n)| \\
|e_y(t_{n+1})| - |e_y(t_n)| &\approx \Delta_1 \cdot |\dot{e}_y(t_n)| - b_2 \cdot |e_y(t_n)| \\
|e_z(t_{n+1})| - |e_z(t_n)| &\approx \Delta_1 \cdot |\dot{e}_z(t_n)| - b_3 \cdot |e_z(t_n)|
\end{aligned} \tag{10}$$

Therefore,

$$\begin{aligned}
&(|e_x(t_{n+1})| - |e_x(t_n)|) + (|e_y(t_{n+1})| - |e_y(t_n)|) + \\
&(|e_z(t_{n+1})| - |e_z(t_n)|) \\
&\approx (\Delta_1 \cdot |\dot{e}_x(t_n)| - k \cdot |e_x(t_n)|) + (\Delta_1 \cdot |\dot{e}_y(t_n)| - \\
&\quad k \cdot |e_y(t_n)|) + (\Delta_1 \cdot |\dot{e}_z(t_n)| - k \cdot |e_z(t_n)|) \\
&= \Delta_1 \cdot (|\dot{e}_x(t_n)| + |\dot{e}_y(t_n)| + |\dot{e}_z(t_n)|) - \\
&\quad b_1 \cdot |e_x(t_n)| - b_2 \cdot |e_y(t_n)| - b_3 \cdot |e_z(t_n)|)
\end{aligned} \tag{11}$$

From (4), we have

$$\begin{aligned}
&|\dot{e}_x(t_n)| \\
&= |-a(t_n) \cdot e_x(t_n) + a(t_n) \cdot e_y(t_n) - (-\Delta a(t_n) \cdot x_2(t_n) + \\
&\quad \Delta a(t_n) \cdot y_2(t_n))| \\
&< |a(t_n)|(|(e_y(t_n)| + |e_x(t_n)|) + |\Delta a(t_n)| \cdot |x_2(t_n) - y_2(t_n)|
\end{aligned} \tag{12}$$

$$\begin{aligned}
&|\dot{e}_y(t_n)| \\
&= |(c(t_n) - a(t_n)) \cdot e_x(t_n) + c(t_n) \cdot e_y(t_n) + x_1(t_n) \cdot z_1(t_n) - \\
&\quad (((\Delta c(t_n) - \Delta a(t_n)) \cdot x_2(t_n) - \Delta c(t_n) \cdot y_2(t_n)) + \\
&\quad x_2(t_n) \cdot z_2(t_n))| \\
&= |(c(t_n) - a(t_n))e_x(t_n) + c(t_n) \cdot e_y(t_n) + \Delta c(t_n) \cdot (x_2(t_n) - \\
&\quad y_2(t_n)) + \Delta a(t_n) \cdot x_2(t_n) + e_x(t_n) \cdot e_z(t_n) + \\
&\quad x_2(t_n) \cdot e_x(t_n) + z_2(t_n) \cdot e_x(t_n)| \\
&< |(c(t_n) - a(t_n)) \cdot e_x(t_n)| + |c(t_n) \cdot e_y(t_n)| + \\
&\quad |\Delta c(t_n) \cdot (x_2(t_n) - y_2(t_n))| + |\Delta a(t_n) \cdot x_2(t_n)| + \\
&\quad |e_x(t_n) \cdot e_z(t_n)| + |x_2(t_n) \cdot e_x(t_n)| + |z_2(t_n) \cdot e_z(t_n)|
\end{aligned} \tag{13}$$

$$\begin{aligned}
&|\dot{e}_z(t_n)| \\
&= |b(t_n) \cdot e_z(t_n) + x_1(t_n) \cdot y_1(t_n) - (\Delta b(t_n) \cdot z_2(t_n) \\
&\quad + x_2(t_n) \cdot y_2(t_n))| \\
&= |b(t_n) \cdot e_z(t_n) - \Delta b(t_n) \cdot z_2(t_n) + (x_2(t_n) + e_x(t_n)) \times \\
&\quad (y_2(t_n) + e_y(t_n)) - x_2(t_n) \cdot y_2(t_n)| \\
&< |b(t_n) \cdot e_z(t_n)| + |\Delta b(t_n) \cdot z_2(t_n)| + |x_2(t_n) \cdot e_y(t_n)| + \\
&\quad |y_2(t_n) \cdot e_x(t_n)| + |e_x(t_n) \cdot e_y(t_n)|
\end{aligned} \tag{14}$$

From (12), (13) and (14), we can derive the following inequation

$$\begin{aligned}
&\Delta_1 \cdot (|\dot{e}_x(t_n)| + |\dot{e}_y(t_n)| + |\dot{e}_z(t_n)|) - b_1 \cdot |e_x(t_n)| - \\
&b_2 \cdot |e_y(t_n)| - b_3 \cdot |e_z(t_n)| \\
&< |e_x(t_n)| \cdot (\Delta_1 \cdot ((1 + \xi_c) \cdot c + \sup(x_2(t_n)) + \sup(y_2(t_n))) - b_1) \\
&\quad + |e_y(t_n)| \cdot (\Delta_1 \cdot ((1 + \xi_a) \cdot a + (1 + \xi_c) \cdot c) + \sup(x_2(t_n)) + \\
&\quad \sup(e_x(t_n)) - b_2) + |e_z(t_n)| \cdot (\Delta_1 \cdot ((1 + \xi_b) \cdot b + \sup(e_x(t_n)) \\
&\quad + \sup(x_2(t_n)) - b_3) + \Delta_1 \cdot (2a \cdot \xi_a \cdot \sup(x_2(t_n) - y_2(t_n)) + \\
&\quad 2c \cdot \xi_c \cdot \sup(x_2(t_n) - y_2(t_n)) + 2a \cdot \xi_a \cdot \sup(x_2(t_n)) + \\
&\quad 2b \cdot \xi_b \cdot \sup(z_2(t_n)))
\end{aligned}$$

$$\begin{aligned}&< (|e_x(t_n)| + |e_y(t_n)| + |e_z(t_n)|)\times \\ &(\Delta_1\cdot \max(\mathrm{m}(\xi_c),\mathrm{n}(\xi_a,\xi_c),\mathrm{o}(\xi_b)) - \min(b_1,b_2,b_3)) + \\ &\Delta_1\cdot C(\xi_a,\xi_b,\xi_c)\end{aligned} \quad (15)$$

Thus we have

$$\begin{aligned}&(|e_x(t_n)| + |e_y(t_n)| + |e_z(t_n)|)\times \\ &(\Delta_1\cdot \max(\mathrm{m}(\xi_c),\mathrm{n}(\xi_a,\xi_c),\mathrm{o}(\xi_b)) - \min(b_1,b_2,b_3)) \\ &+\Delta_1\cdot C(\xi_a,\xi_b,\xi_c) < 0\end{aligned}$$

only if

$$\Delta_1\cdot \max(\mathrm{m}(\xi_c),\mathrm{n}(\xi_a,\xi_c),\mathrm{o}(\xi_b)) - \min(b_1,b_2,b_3) < 0$$

Hence

$$\begin{aligned}&(|e_x(n\cdot\Delta_1)| + |e_y(n\cdot\Delta_1)| + |e_z(n\cdot\Delta_1)|)\times \\ &(\Delta_1\cdot \max(\mathrm{m}(\xi_c),\mathrm{n}(\xi_a,\xi_c),\mathrm{o}(\xi_b)) - \min(b_1,b_2,b_3)) + \\ &\Delta_1\cdot C(\xi_a,\xi_b,\xi_c) \\ &< \varepsilon\cdot(\Delta_1\cdot \max(\mathrm{m}(\xi_c),\mathrm{n}(\xi_a,\xi_c),\mathrm{o}(\xi_b)) - \min(b_1,b_2,b_3)) + \\ &\Delta_1\cdot C(\xi_a,\xi_b,\xi_c)\end{aligned} \quad (16)$$

If the length of the impulse interval satisfied

$$\Delta_1 < \frac{\min(b_1,b_2,b_3)}{\max(\mathrm{m}(\xi_c),\mathrm{n}(\xi_a,\xi_c),\mathrm{o}(\xi_b)) + C(\xi_a,\xi_b,\xi_c)/\varepsilon},$$

based on the inequation (16), we have

$$\begin{aligned}&(|e_x((n+1)\cdot\Delta_1)| - |e_x(n\cdot\Delta_1)|) + (|e_y((n+1)\cdot\Delta_1)| - |e_y(n\cdot\Delta_1)|) \\ &+(|e_z((n+1)\cdot\Delta_1)| - |e_z(n\cdot\Delta_1)|) < 0\end{aligned}$$

Thus condition 1 of **Lemma** 3 is satisfied.

Case 2. $\|e(1\cdot\Delta_1)\| < \varepsilon$

In this case, the error system between the transmitter and the receiver can be written as

$$\begin{cases}\dot{\mathbf{e}} = ((\mathbf{A}+\Delta\mathbf{A}_1(t))\mathbf{X}_1 - (\mathbf{A}+\Delta\mathbf{A}_2(t))\mathbf{X}_2) + (\varphi(\mathbf{X}_1) - \varphi(\mathbf{X}_2)) & t \neq \tau_i \\ \mathbf{e}(\tau_i) = \mathbf{e}(\tau_i^-) - \mathbf{B}(\mathbf{e}(\tau_i^-) - (C(\mathbf{X}_2(\tau_i^-),\mathbf{EC}(\tau_i^-)) - Q(\mathbf{X}_1(\tau_i^-))) & i = p+1, p+2,\ldots\end{cases}$$

Based on (10) and (12), (13), (14), and Lemma 1 we have

$$\begin{aligned}&|e_x(t_{n+1})| + |e_y(t_{n+1})| + |e_y(t_{n+1})| \\ &= (1-b_1)\cdot|e_x(t_n)| + (1-b_2)\cdot|e_y(t_n)| + (1-b_3)\cdot|e_z(t_n)|) + \\ &\Delta_1\cdot(|\dot{e}_x(t_n)| + |\dot{e}_y(t_n)| + |\dot{e}_z(t_n)|) \\ &< (1-\min(b_1,b_2,b_3) + \Delta_1\cdot \max(\mathrm{m}(\xi_c),\mathrm{n}(\xi_a,\xi_c),\mathrm{o}(\xi_b)))\cdot\varepsilon + \\ &\Delta_1\cdot C(\xi_a,\xi_b,\xi_c)\end{aligned} \quad (17)$$

Hence, if the length of the impulse interval satisfied

$$\Delta_1 < \frac{\min(b_1,b_2,b_3)}{\max(\mathrm{m}(\xi_c),\mathrm{n}(\xi_a,\xi_c),\mathrm{o}(\xi_b)) + C(\xi_a,\xi_b,\xi_c)/\varepsilon},$$

based on the inequation (14), we have

$$|e_x((n+1)\cdot\Delta_1)| + |e_y((n+1)\cdot\Delta_1)| + |e_y((n+1)\cdot\Delta_1)| < \varepsilon$$

Thus condition 2 of **Lemma** 3 is satisfied.

In summary, if the length of the impulsive interval satisfied:

$$\Delta_1 < \frac{\min(b_1,b_2,b_3)}{\max(\mathrm{m}(\xi_c),\mathrm{n}(\xi_a,\xi_c),\mathrm{o}(\xi_b)) + C(\xi_a,\xi_b,\xi_c)/\varepsilon}$$

then ∃ t_0, such that for $\forall t > t_0$, $|e_x(t)| < \varepsilon$, $|e_y(t)| < \varepsilon$, $|e_z(t)| < \varepsilon$.

Remark 1: Theorem 1 only corresponds to Chen system. However, the method can also been used by other chaotic systems.

Remark 2: Note that in case 2, only one bit error correcting signal is transmitted through the channel for each synchronization impulse.

V. NUMERICALEXAMLPE

In this section, to verify the effectiveness of proposed scheme, we implement synchronization of the following Chen chaotic model by using the circuit simulation software Cadence. Suppose that the Chen model at the transmitter and the receiver are perturbed with parametric uncertainties

$$\begin{aligned}&\Delta a_1(t) = \xi_a\cdot a\cdot\sin(t), \Delta a_2(t) = \xi_a\cdot a\cdot\cos(t) \quad a = 35 \\ &\Delta b_1(t) = \xi_b\cdot b\cdot\sin(t), \Delta b_2(t) = \xi_b\cdot b\cdot\cos(t) \ , \ b = 8/3 \\ &\Delta c_1(t) = \xi_c\cdot c\cdot\sin(t), \Delta c_2(t) = \xi_c\cdot c\cdot\cos(t) \quad \mathrm{c}=28\end{aligned}$$

$$\begin{aligned}&\varepsilon = 0.01 \\ &\xi_a = \xi_b = \xi_c = 0.05\end{aligned}, \ \mathbf{B} = \begin{bmatrix}0.9 & & \\ & 0.8 & \\ & & 0.7\end{bmatrix}$$

According to the result shown in[19], Chen system is bounded. There exist a $0<\eta<1$ such that

$$\eta^4 + \frac{2(b+c)}{c}\eta^3 + \frac{2(b-c)}{c}\eta - 1 = 0$$

and we have

$$x^2 + y^2 + (z-c)^2 \le R^2 \quad (18)$$

where *x*, *y*, *z* are the state of Chen system and

$$R^2 = \frac{(a+c)(a-c)^2(1+\eta)^2}{16a\eta^2}[\frac{b}{c}(1-\eta)^2 + (1+\eta)^2] + \frac{4(a^2-c^2)(a^2+c^2-bc) + c^2(2a-b)^2}{4a(b+c)}$$

Thus

$$\begin{aligned}&\min(b_1,b_2,b_3) = 0.7 \\ &\max(\mathrm{m}(\xi_c),\mathrm{n}(\xi_a,\xi_c),\mathrm{o}(\xi_b)) = 190.2426 \\ &C(\xi_a,\xi_b,\xi_c) = 531.8080\end{aligned}$$

According to **Theorem** 1,

$$\Delta_1 < 1.3116\times10^{-5}$$
$$\delta \ge 0.02$$

We choose $\Delta_1 = 1.2\times10^{-5}$ as the impulse interval and $\delta = 0.03125$ as the quantization step of the A/D module in the simulation. The simulation results are illustrated in Fig.3 and Fig 4. As it shown in Fig.3, the error between two systems is decreasing while the error is bigger than the prescribed synchronization threshold. And in Fig.4, when the error is smaller than the synchronization threshold, the error keeps oscillating within the prescribed bound.

VI. CONCLUSION

In this paper, we present a chaotic synchronization scheme based on impulsive stability theory. The process of synchronization is divided into two stages. In the first stage, the magnitude of the synchronization error approach to the pre-specified bound from the initial value. In the second stage, the magnitude of the synchronization error stay within the pre-specified bound. And in this stage, only one bit information is needed to transmit from the transmitter to the receiver for each synchronization impulse. As a result, the amount of information transmitted between the two systems is reduced significantly. The quantitative relationship between the prescribed synchronization threshold, the impulse interval, the bounds of uncertainties and the impulse intensity is derived. Furthermore, a numerical example is given to illustrate the effectiveness of the proposed scheme.

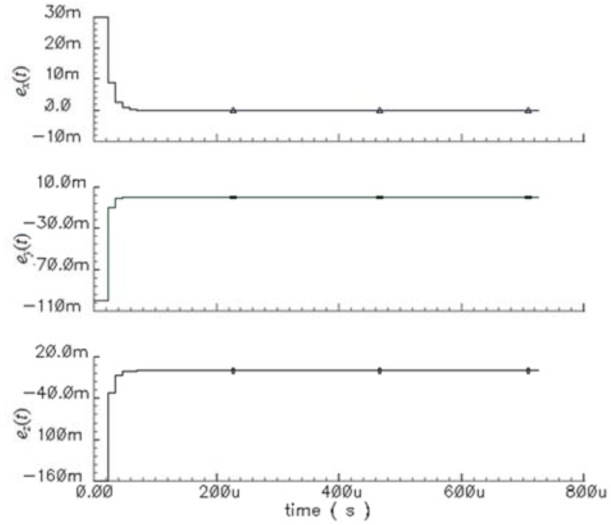

Figure 3. Simulation results for the synchronization of Chen system: The decreasing process of synchronization error.

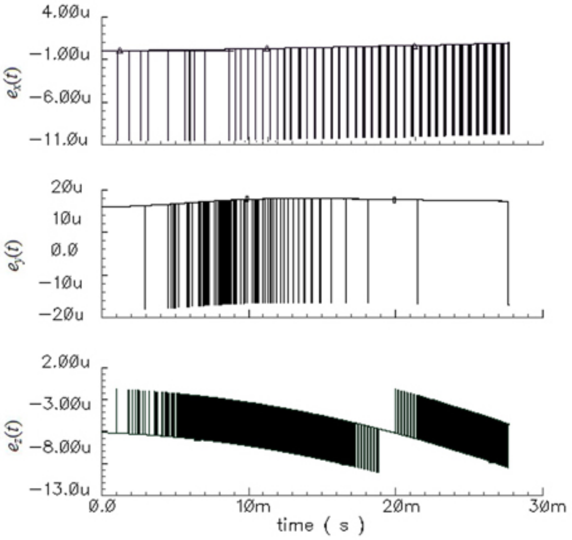

Figure 4. Simulation results for the synchronization of Chen system: The oscillation process of synchronization error.

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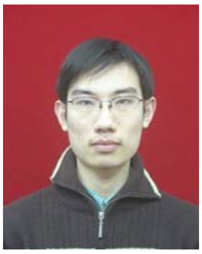

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