

Dynamics in a Coupled FHN Model with Two Different Delays

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Abstract—In this paper, a coupled FHN model with two different delays is investigated. The local stability and the existence of Hopf bifurcation for the system are analyzed. The effect of two different delays on dynamical behavior is discussed. Simulation results are presented to support theoretical analysis. Finally, main conclusions are included.

Index Terms—FHN model, delay, stability, Hopf bifurcation, periodic solution

I. INTRODUCTION

The FHN model with cubic nonlinearity has been obtained from a simplified Hodgkin-Huxley(HH) neuron model [1-3]. Its complete topological and qualitative investigation has been carried out [4] and a rich variety of nonlinear phenomena such as hard oscillation, separatrix loops, bifurcations for equilibrium, resonance phenomena and limit cycles has been observed [5-11]. Since time delays always occur in the signal transmission for real neurons, Dhamala et al. [12] made some theoretical discussion on coupled time-delay oscillators. Nikola and Dragana [13] and Nikola et al. [14] had dealt with the bifurcation and synchronization in coupled identical neurons with delayed coupling. Recently, Wang et al. [15] has numerically investigated the bifurcation and synchronization of the following delayed coupled FHN system with synaptic connection

$$\begin{cases} \dot{V}_1(t) = -V_1^3 + aV_1 - W_1 + C_1 \tanh(V_2(t - \tau)), \\ \dot{W}_1(t) = V_1 - b_1W_1, \\ \dot{V}_2(t) = -V_2^3 + aV_2 - W_2 + C_2 \tanh(V_1(t - \tau)), \\ \dot{W}_2(t) = V_2 - b_2W_2, \end{cases} \quad (1)$$

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where $V_1(t), V_2(t)$ represent the transmembrane voltage, $W_1(t), W_2(t)$ should model the time dependence of several physical quantities related to electrical variables. $a, b_i, C_i (i = 1, 2)$ are positive constants, τ represents time delay, i.e., the function which describes the influence of the i -th unit on the j -th unit at the time t depends on the state of the i -th unit at some earlier time $t - \tau$. The more detailed meaning of the coefficients of system (1), one can see [15].

In order to describe model (1) more reasonable, Fan and Hong [13] modified (1) as the following form

$$\begin{cases} \dot{V}_1(t) = -V_1^3 + aV_1 - W_1 + C_1 \tanh(V_2(t - \tau_1)), \\ \dot{W}_1(t) = V_1 - b_1W_1, \\ \dot{V}_2(t) = -V_2^3 + aV_2 - W_2 + C_2 \tanh(V_1(t - \tau_2)), \\ \dot{W}_2(t) = V_2 - b_2W_2 \end{cases} \quad (2)$$

and considered the Hopf bifurcation properties of system (2).

It shall be pointed out that Wang et al. [15] analyzed the Hopf bifurcation under the assumption $\tau_1 = \tau_2 = \tau$, Fan and Hong [13] made a discussion on the Hopf bifurcation of system (2) under the the condition $\tau_1 + \tau_2 = \tau$, but in most cases, $\tau_1 \neq \tau_2$, the two different delays have different effect on the dynamical behavior of system (2). Considering this factor, we further investigate the model (2) with $\tau_1 \neq \tau_2$ as a complementarity.

The main goal of this paper is to study the stability, the local Hopf bifurcation for system (2). It is shown that different delays have different effect on the dynamical behavior of system involved. Recently, although a great deal of research has been devoted to this topic [17-20], to the best of our knowledge, there are few papers that consider what different time delays have effect on the dynamical behavior of system. We believe that it is the first time to deal with the research on Hopf bifurcation for model (2) under the assumption $\tau_1 \neq \tau_2$.

The remainder of the paper is organized as follows. In Section 2, we investigate the stability of the zero

equilibrium and the occurrence of local Hopf bifurcations. In Section 3, numerical simulations are carried out to illustrate the validity of the main results. Some main conclusions are drawn in Section 4.

II. STABILITY OF THE EQUILIBRIUM AND LOCAL HOPF BIFURCATIONS

In this section, we shall focus on the stability of the zero equilibrium and the existence of local Hopf bifurcations.

Since time delay does not change the equilibrium of system, then the delayed coupled FHN model (2) has an equilibrium point $E(0, 0, 0, 0)$.

The linearization of Eq. (2) at $E(0, 0, 0, 0)$ is given by

$$\begin{cases} \dot{V}_1(t) = aV_1 - W_1 + C_1V_2(t - \tau_1), \\ \dot{W}_1(t) = V_1 - b_1W_1, \\ \dot{V}_2(t) = aV_2 - W_2 + C_2V_1(t - \tau_2), \\ \dot{W}_2(t) = V_2 - b_2W_2. \end{cases} \quad (3)$$

The characteristic equation of system (3) is

$$\lambda^4 + m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + (n_1\lambda + n_0)e^{-\lambda(\tau_1 + \tau_2)} = 0, \quad (4)$$

where

$$\begin{cases} m_0 = a^2b_1b_2 - a(b_1 + b_2), \\ m_1 = a^2(b_1 + b_2) - 2ab_1b_2 + b_1 + b_2 - 2a, \\ m_2 = b_1b_2 + a^2 - 2a(b_1 + b_2) + 2, \\ m_3 = b_1 + b_2 - 2a, \\ n_0 = b_1c_1c_2, n_1 = c_1c_2. \end{cases}$$

In the sequel, we will discuss the distribution of roots of the transcendental equation (4). Now we consider three cases.

Case (a). $\tau_1 = \tau_2 = 0$, (4) becomes

$$\lambda^4 + m_3\lambda^3 + m_2\lambda^2 + (m_1 + n_1)\lambda + m_0 + n_0 = 0. \quad (5)$$

All roots of (5) have a negative real part if the following condition

$$(H1) \quad m_3 > 0, m_2m_3 > m_1 + n_1,$$

$$m_2m_3(m_1 + n_1) > m_3^2(m_0 + n_0) + (m_1 + n_1)^2$$

holds. Then the equilibrium point $E(0, 0, 0, 0)$ is locally asymptotically stable if the condition (H1) holds.

Case (b). $\tau_2 = 0, \tau_1 > 0$, (4) becomes

$$\lambda^4 + m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + (n_1\lambda + n_0)e^{-\lambda\tau_1} = 0. \quad (6)$$

For $\omega > 0$, $i\omega$ is a root of (6), then

$$\begin{cases} n_1\omega \sin \omega\tau_2 + n_0 \cos \omega\tau_2 = m_2\omega^2 - \omega^4 - m_0, \\ n_1\omega \cos \omega\tau_2 - n_0 \sin \omega\tau_2 = m_3\omega^3 - m_1\omega. \end{cases} \quad (7)$$

Then

$$\omega^8 + p\omega^6 + q\omega^4 + u\omega^2 + v = 0, \quad (8)$$

where $p = m_3^2 - 2m_2, q = m_2^2 + 2m_0 - 2m_1m_3, u = m_1^2 - 2m_0m_2 - n_1^2, v = m_0^2 - n_0^2$.

Let $z = \omega^2$, then (8) takes the form

$$z^4 + pz^3 + qz^2 + uz + v = 0. \quad (9)$$

Since the form of (9) is identical to those of (6) in Fan and Hong [16] and (9) in Li and Wei [21], then we can obtain Lemma 2.1 and Lemma 2.2 analogously. The proofs are omitted.

Lemma 2.1. [16,21] *If $v < 0$, then (9) has at least one positive root.*

Denote

$$h(z) = z^4 + pz^3 + qz^2 + uz + v. \quad (10)$$

Then

$$h'(z) = 4z^3 + 3pz^2 + 2qz + u. \quad (11)$$

Set

$$4z^3 + 3pz^2 + 2qz + u = 0. \quad (12)$$

Let $y = z + \frac{p}{4}$, then (12) becomes

$$y^3 + p_1y + q_1 = 0, \quad (13)$$

where $p_1 = \frac{q}{2} - \frac{3}{16}p^2, q_1 = \frac{p^3}{32} - \frac{pq}{8} + \frac{u}{4}$.

Define

$$\begin{cases} \Delta = \left(\frac{q_1}{2}\right)^2 + \left(\frac{p_1}{3}\right)^3, \varepsilon = \frac{-1+i\sqrt{3}}{2}, \\ y_1 = \sqrt[3]{-\frac{q_1}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q_1}{2} - \sqrt{\Delta}}, \\ y_2 = \sqrt[3]{-\frac{q_1}{2} + \sqrt{\Delta}\varepsilon} + \sqrt[3]{-\frac{q_1}{2} - \sqrt{\Delta}\varepsilon^2}, \\ y_3 = \sqrt[3]{-\frac{q_1}{2} + \sqrt{\Delta}\varepsilon^2} + \sqrt[3]{-\frac{q_1}{2} - \sqrt{\Delta}\varepsilon}. \end{cases}$$

Let $z_i = y_i - \frac{p}{4} (i = 1, 2, 3)$.

Lemma 2.2. [16,21] *Suppose that $v \geq 0$, then we have the following results.*

(i) *If $\Delta \geq 0$, then (9) has positive roots if and only if $z_1 > 0$ and $h(z_1) < 0$.*

(ii) *If $\Delta < 0$, then (9) has positive roots if and only if there exists at least one $z^* \in \{z_1, z_2, z_3\}$ such that $z^* > 0$ and $h(z^*) \leq 0$.*

Suppose that (9) has positive roots. Without loss of generality, we assume that it has four positive roots, denoted by $z_k^* (k = 1, 2, 3, 4)$. Then (8) has four positive roots $\omega_k = \sqrt{z_k^*} (k = 1, 2, 3, 4)$. In view of (7), we get $\tau_{1k}^{(j)} =$

$$\frac{1}{\omega_k} \left\{ \arccos \left[\frac{m_2\omega_k^2 - m_0}{n_0^2 + n_1^2\omega_k^2} \right] + 2j\pi \right\}, \quad (14)$$

where $k = 1, 2, 3, 4; j = 0, 1, 2, 3, \dots$. Then $\pm i\omega_k$ are a pair of purely imaginary roots of (6) with $\tau = \tau_{1k}^{(j)}$. Obviously, the sequence $\{\tau_{1k}^{(j)}\}_{j=0}^{+\infty}$ is increasing and $\lim_{j \rightarrow +\infty} \tau_{1k}^{(j)} = +\infty (k = 1, 2, 3, 4)$.

For convenience, we let

$$\cup_{k=1}^4 \{\tau_{1k}^{(j)}\}_{j=0}^{+\infty} = \{\tau_{1i}\}_{i=0}^{+\infty}$$

such that

$$\tau_{10} < \tau_{11} < \tau_{12} < \dots < \tau_{1i} < \dots, \quad (15)$$

where

$$\tau_{1_0} = \min\{\tau_{1_1}^{(0)}, \tau_{1_2}^{(0)}, \tau_{1_3}^{(0)}, \tau_{1_4}^{(0)}\}.$$

Applying Lemma 2.1 and Lemma 2.2, we have the following results.

Lemma 2.3. *Assume that (H1) holds, then we have the following results.*

(i) *If one of the following holds: (a) $v < 0$; (b) $v \geq 0, m_0 \geq 0, z_1 > 0$ and $h(z_1) \leq 0$; (c) $v \geq 0, m_0 < 0$, and there exists a $z^* \in \{z_1, z_2, z_3\}$ such that $z^* > 0$ and $h(z^*) \leq 0$, then all roots of (6) have negative real parts when $\tau \in [0, \tau_{1_0})$.*

(ii) *If the conditions (a)-(c) of (i) are not satisfied, then all roots (6) have negative real parts for all $\tau_1 \geq 0$.*

Let $\lambda(\tau_1) = \alpha(\tau_1) + i\omega(\tau_1)$ be a root of (6) near $\tau_1 = \tau_{1_k}$ and $\alpha(\tau_{1_k}) = 0, \omega(\tau_{1_k}) = \omega_0$. According to Lemma 2.3 in Ruan and Wei [22], Lemma 2.4 in Li and Wei [21], Lemma 2.5 in Hu and Huang [23] and Lemma 2.5 in Fan and Hong [16], we have the following conclusions.

Lemma 2.4. *Suppose $h'(z_k^*) \neq 0$, where $h(z)$ is defined by (10). If $\tau_1 = \tau_{1_k}^{(j)}$, then $\pm i\omega_k$ are a pair of simple purely imaginary roots of Eq. (6). Moreover,*

$$\left. \frac{d(\text{Re}\lambda(\tau_1))}{d\tau_1} \right|_{\tau_1=\tau_{1_k}^{(j)}} \neq 0,$$

and the sign of $\left. \frac{d(\text{Re}\lambda(\tau_1))}{d\tau_1} \right|_{\tau_1=\tau_{1_k}^{(j)}}$ is consistent with that of $h'(z_k^*)$.

In order to obtain our main results, we assume that

$$(H2) \quad h'(z_k^*) > 0.$$

Case (c). $\tau_2 > 0, \tau_1 > 0$. We consider Eq. (4) with τ_1 in its stable interval. Regarding τ_2 as a parameter. Without loss of generality, we consider system (2) under the assumptions (H1) and (H2). Let $i\omega(\omega > 0)$ be a root of (4), then we can obtain

$$\omega^8 + k_1\omega^6 + k_2\omega^4 + k_3\omega^2 + k_4 = 0, \quad (16)$$

where

$$\begin{aligned} k_1 &= m_3^2 - 2m_2, k_2 = m_2^2 + 2m_0 - 2m_3n_1, \\ k_3 &= n_1^2 - 2m_0m_2 - n_1^2 \sin\omega\tau_1 - n_1^2 \cos^2\omega\tau_1, \\ k_4 &= m_0^2 - n_0^2. \end{aligned}$$

Denote

$$H(\omega) = \omega^4 + k_1\omega^3 + k_2\omega^2 + k_3\omega + k_4. \quad (17)$$

Assume that

$$(H3) \quad |m_0| < |n_0|.$$

It is easy to check that $H(0) < 0$ if (H3) holds and $\lim_{\omega \rightarrow +\infty} H(\omega) = +\infty$. We can obtain that (16) has finite positive roots $\omega_1, \omega_2, \dots, \omega_n$. For every fixed $\omega_i, i = 1, 2, 3, \dots, k$, there exists a sequence $\{\tau_{2_i}^j | j = 1, 2, 3, \dots\}$, such that (16) holds. When $\tau_2 = \tau_{2_i}^j$, Eq.

(4) has a pair of purely imaginary roots $\pm i\omega^*$ for $\tau_1 \in [0, \tau_{1_0})$. In the following, we assume that

$$(H4) \quad \left[\frac{d(\text{Re}\lambda)}{d\tau_2} \right]_{\lambda=i\omega^*} \neq 0.$$

In view of the general Hopf bifurcation theorem for FDEs in Hale [24], we have the following result on the stability and Hopf bifurcation in system (2).

Theorem 2.1. *For system (2), we have the following results.*

(i) *Assume that $\tau_2 = 0$ and (H1) – (H2) are fulfilled, then system (2) is asymptotically stable for $\tau_1 \in [0, \tau_{1_0})$ and unstable for $\tau_1 > \tau_{1_0}$.*

(ii) *Assume that (H1) – (H4) are satisfied and $\tau_1 \in [0, \tau_{1_0})$, then system (2) undergoes a Hopf bifurcation at the zero equilibrium $E(0, 0, 0, 0)$ when $\tau_2 = \tau_{2_i}^j$.*

III. NUMERICAL EXAMPLES

In order to verify the theoretical predations of this paper, numerical simulations are carried out in this section. We consider the following system:

$$\begin{cases} \dot{V}_1(t) = -V_1^3 + 0.05V_1 - W_1 \\ \quad + 0.225 \tanh(V_2(t - \tau_1)), \\ \dot{W}_1(t) = V_1 - 1.28W_1, \\ \dot{V}_2(t) = -V_2^3 + 0.05V_2 - W_2 \\ \quad + 0.225 \tanh(V_1(t - \tau_2)), \\ \dot{W}_2(t) = V_2 - 0.08W_2. \end{cases} \quad (18)$$

Obviously, system (18) has an equilibrium $E(0, 0, 0, 0)$. When $\tau_2 = 0$, then we can easily check that (H1)-(H4) hold true. Let $j = 0$ and by Matlab 7.0, we get $\omega_0 \approx 0.5874, \tau_{1_0} \approx 3.8$. Thus the zero equilibrium $E(0, 0, 0, 0)$ is asymptotically stable for $\tau_1 < \tau_{1_0} \approx 3.8$ and unstable for $\tau_1 > \tau_{1_0} \approx 3.8$ which is shown in Fig. (1)-Fig. (10). When $\tau_1 = \tau_{1_0} \approx 3.8$, Eq. (18) undergoes a Hopf bifurcation around the zero equilibrium $E(0, 0, 0, 0)$, i.e., a small amplitude periodic solution occurs near $E(0, 0, 0, 0)$ when $\tau_2 = 0$ and τ_1 is close to $\tau_{1_0} = 3.8$ which can be illustrated in Fig. (11)-Fig. (20).

Let $\tau_1 = 3 \in (0, 3.8)$ and regard τ_2 as a parameter. We get $\tau_{2_0} \approx 0.2$. It is found that the zero equilibrium is asymptotically stable when $\tau_2 > \tau_{1_0}$. It can be illustrated by the numerical simulations (see Fig. (31)-Fig. (40)) The zero equilibrium $E(0, 0, 0, 0)$ is unstable when $\tau_2 < \tau_{2_0}$. A Hopf bifurcation will occurs around the zero equilibrium $E(0, 0, 0, 0)$ when $\tau_{2_0} \approx 0.2$, i.e., a family of periodic solutions bifurcate from the zero equilibrium $E(0, 0, 0, 0)$ (see Fig. (21)-Fig. (30)).

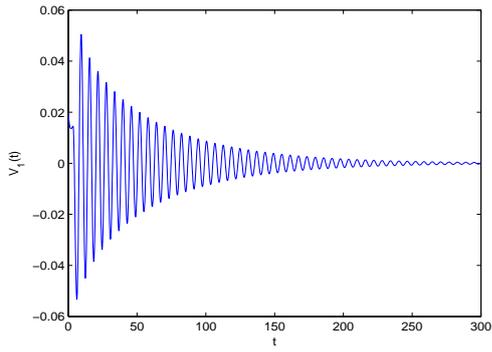


Fig. (1)

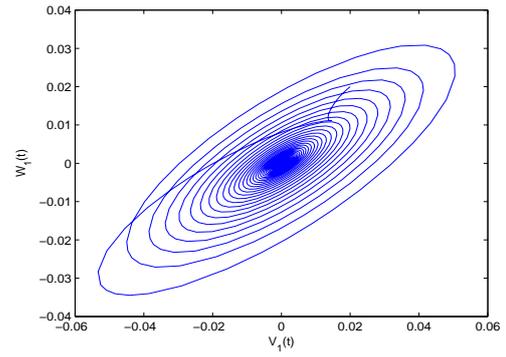


Fig. (5)

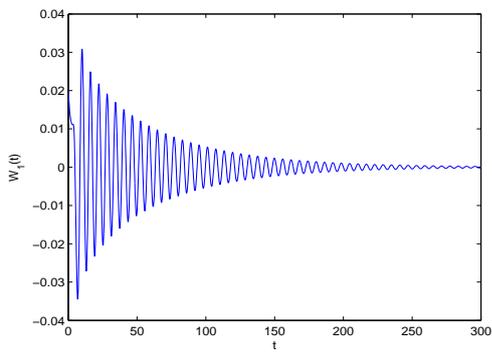


Fig. (2)

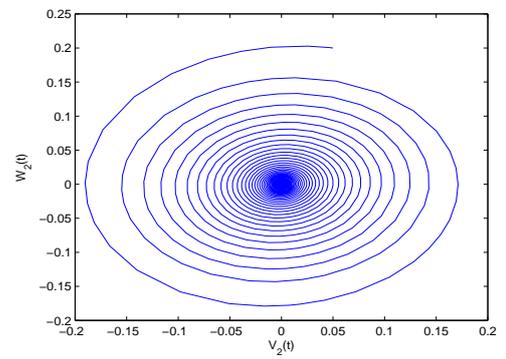


Fig. (6)

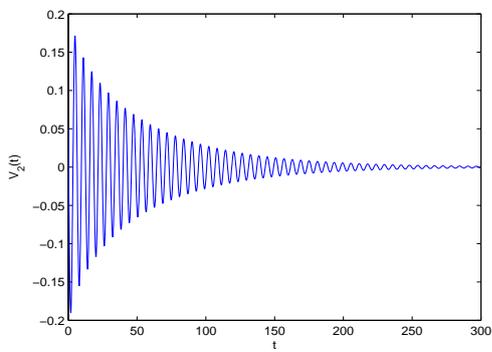


Fig. (3)

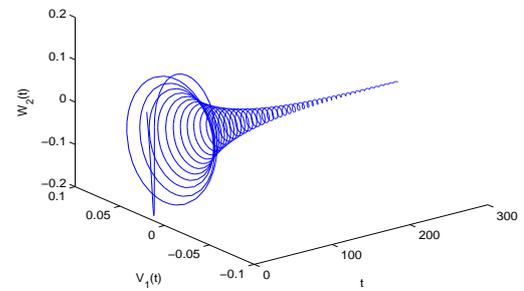


Fig. (7)

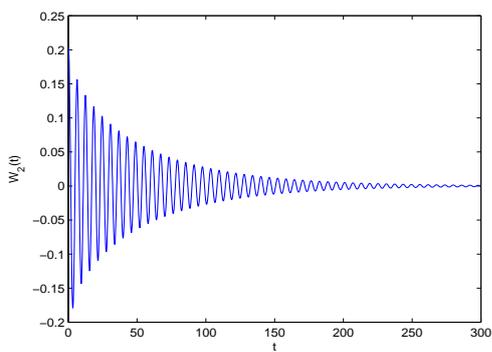


Fig. (4)

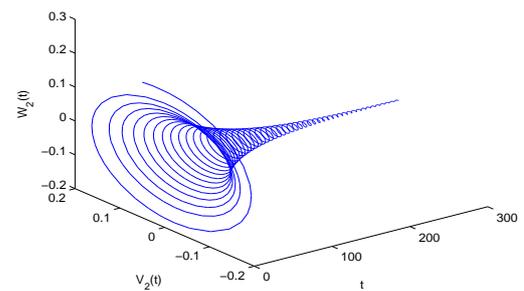


Fig. (8)

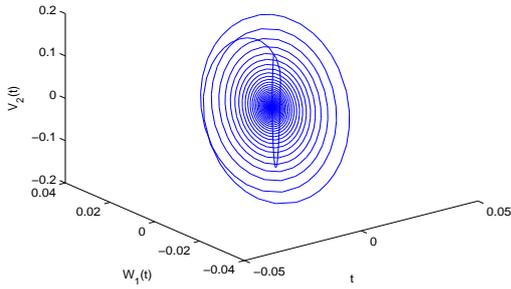


Fig. (9)

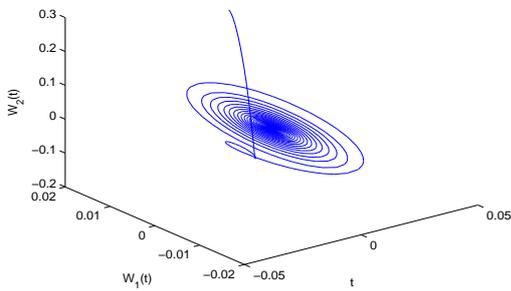


Fig. (10)

Fig. (1)-Fig. (10). Dynamical behavior of system (18) with $\tau_2 = 0, \tau_1 = 3.5 < \tau_{1_0} \approx 3.8$. The zero equilibrium $E(0, 0, 0, 0)$ is asymptotically stable. The initial value is $(0.02, 0.02, 0.05, 0.2)$.

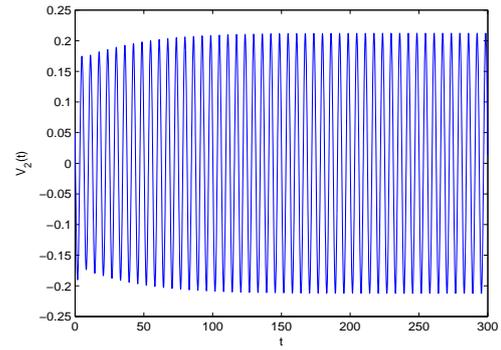


Fig. (13)

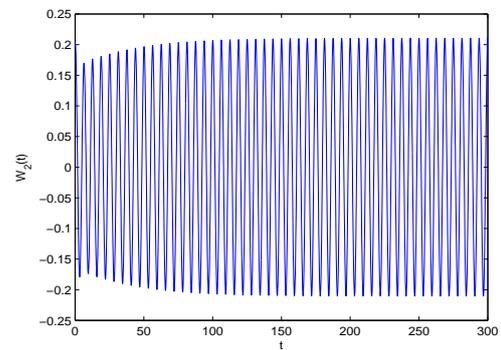


Fig. (14)

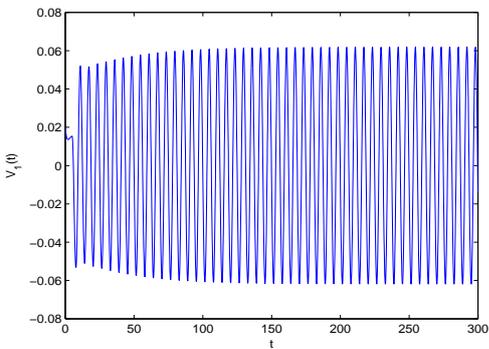


Fig. (11)

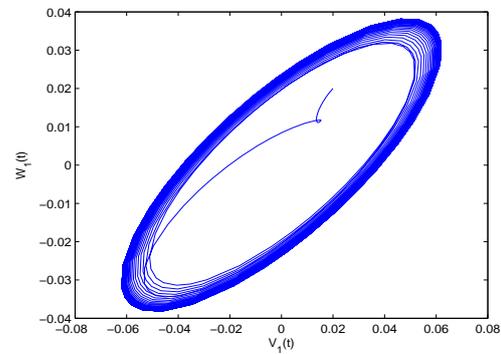


Fig. (15)

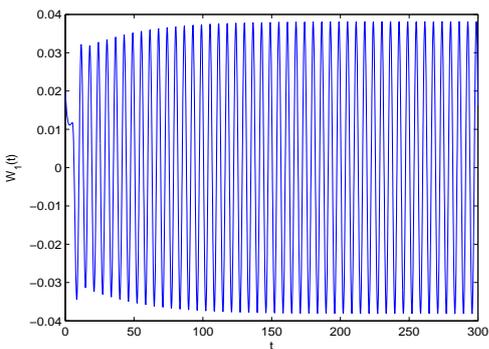


Fig. (12)

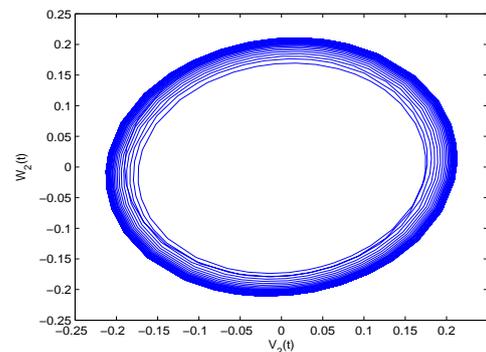


Fig. (16)

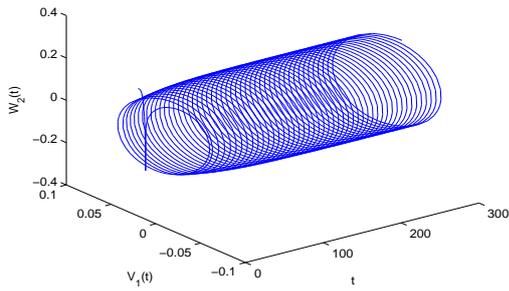


Fig. (17)

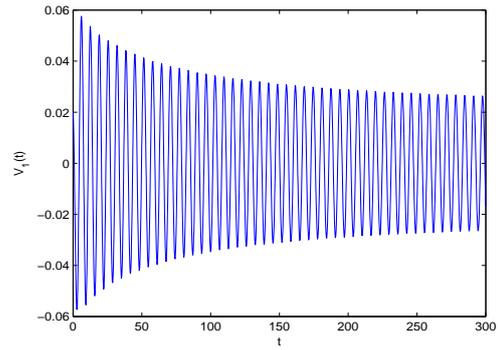


Fig. (21)

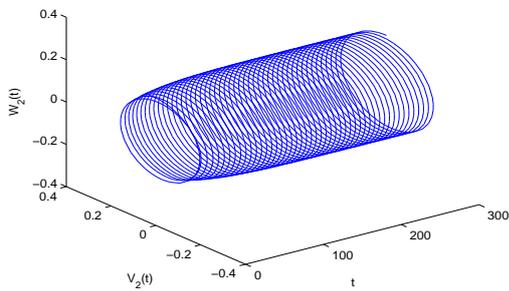


Fig. (18)

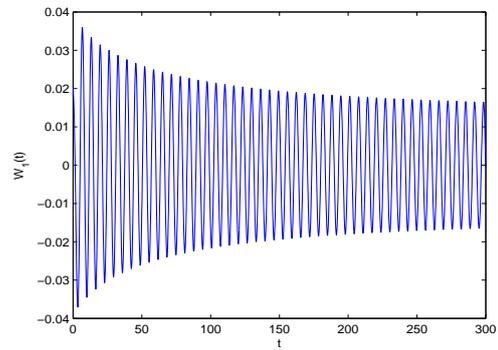


Fig. (22)

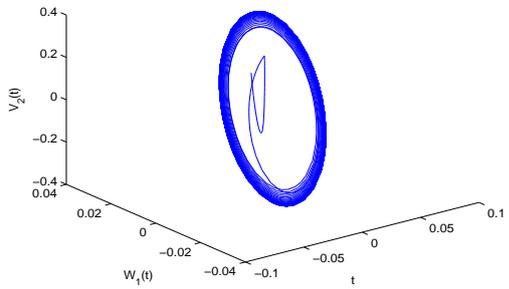


Fig. (19)

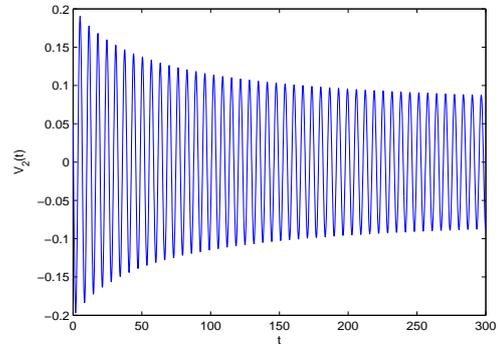


Fig. (23)

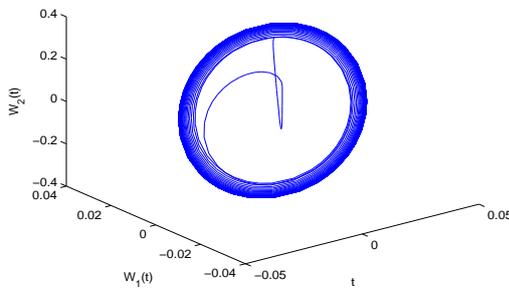


Fig. (20)

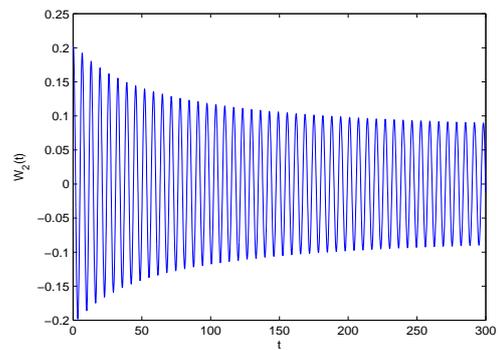


Fig. (24)

Fig. (11)-Fig. (20). Dynamical behavior of system (18) with $\tau_2 = 0, \tau_1 = 5 > \tau_{1_0} \approx 3.8$. The Hopf bifurcation occurs from the zero equilibrium $E(0, 0, 0, 0)$. The initial value is $(0.02, 0.02, 0.05, 0.2)$.

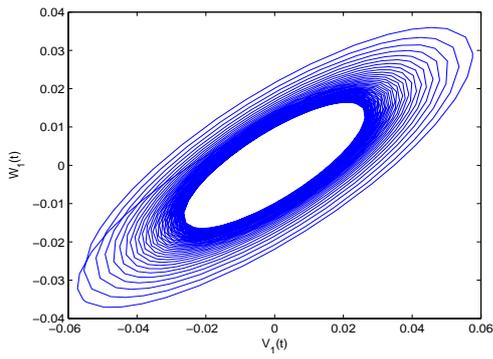


Fig. (25)

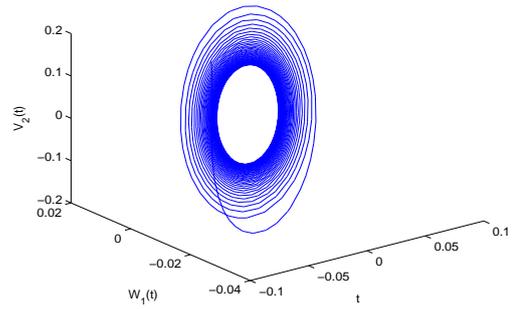


Fig. (29)

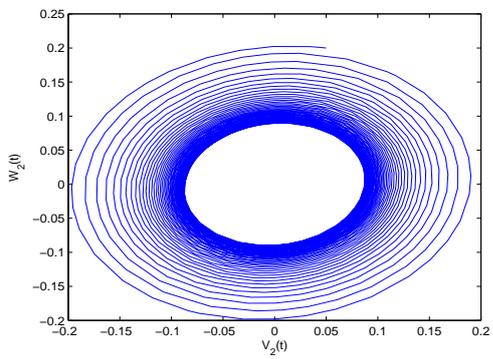


Fig. (26)

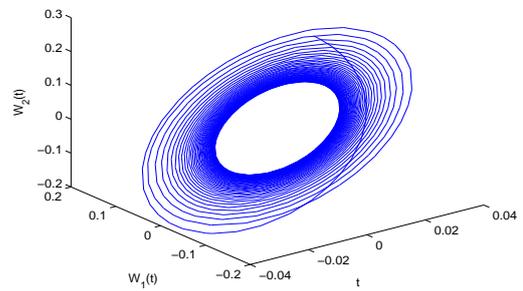


Fig. (30)

Fig. (21)-Fig. (30). Dynamical behavior of system (18) with $\tau_1 = 3, \tau_2 = 0.01 < \tau_{2_0} \approx 0.2$. The Hopf bifurcation occurs from the zero equilibrium $E(0, 0, 0, 0)$. The initial value is $(0.02, 0.02, 0.05, 0.2)$.

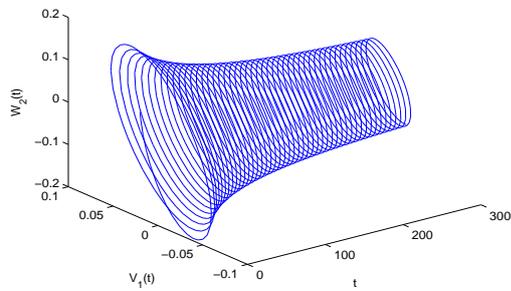


Fig. (27)

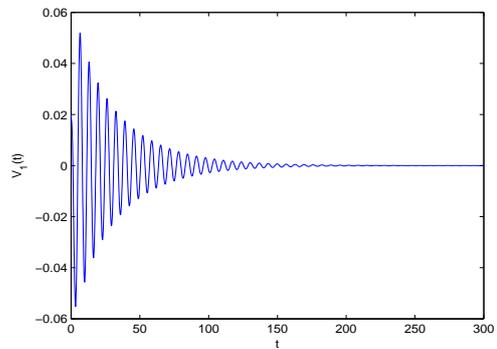


Fig. (31)

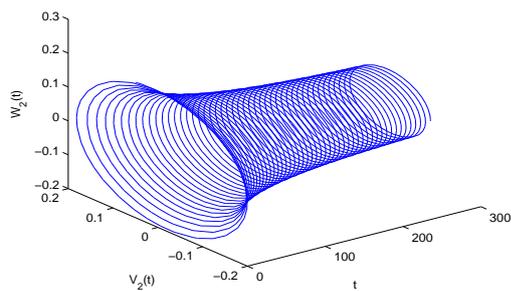


Fig. (28)

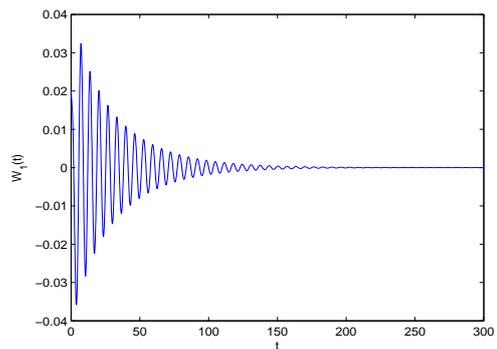


Fig. (32)

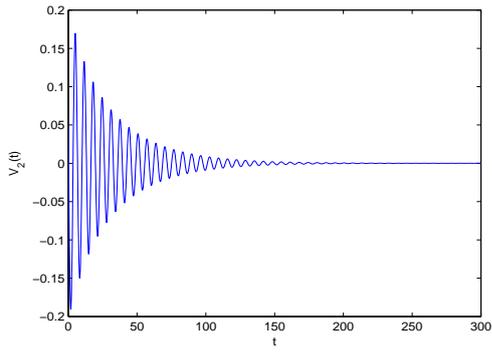


Fig. (33)

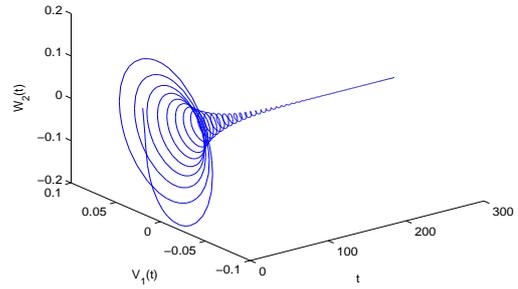


Fig. (37)

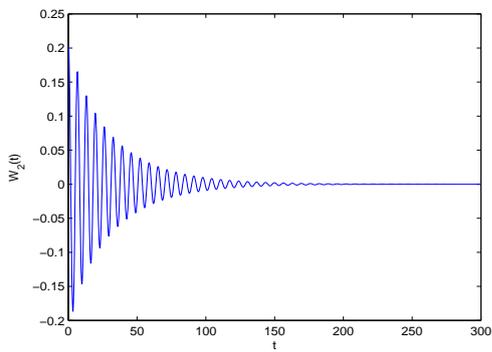


Fig. (34)

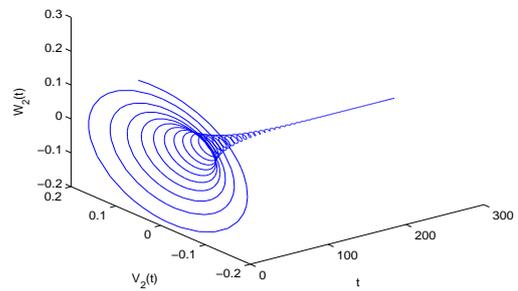


Fig. (38)

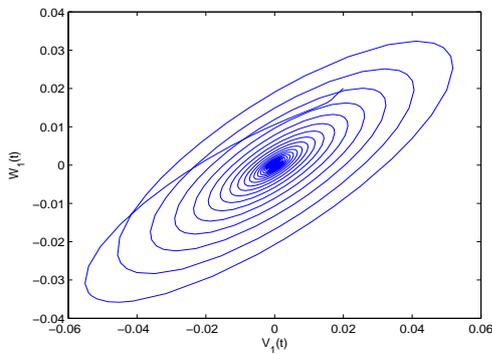


Fig. (35)

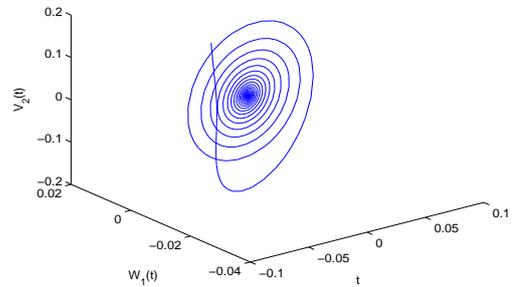


Fig. (39)

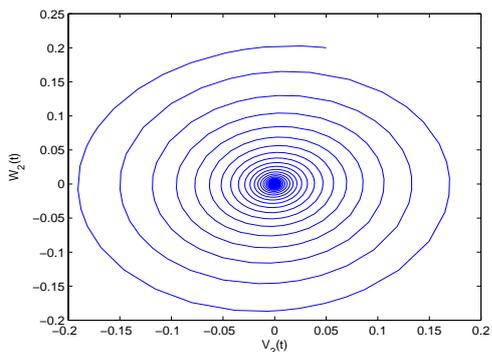


Fig. (36)

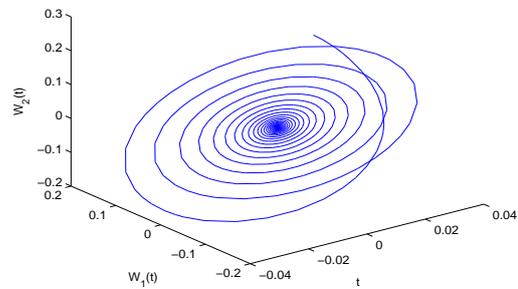


Fig. (40)

Fig. (31)-Fig. (40). Dynamical behavior of system (18) with $\tau_1 = 3, \tau_2 = 0.5 > \tau_{20} \approx 0.2$. The zero equilibrium $E(0, 0, 0, 0)$ is asymptotically stable. The initial value is $(0.02, 0.02, 0.05, 0.2)$.

IV. CONCLUSIONS

In this paper, we have dealt with the local stability of the zero equilibrium $E(0, 0, 0, 0)$ and local Hopf bifurcation of a coupled FHN model with two different delays. We have found that if $\tau_2 = 0$ and (H1)-(H2) are satisfied, then system (2) is asymptotically stable for $\tau_1 \in [0, \tau_{10})$ and unstable for $\tau_1 > \tau_{10}$. If (H1)-(H4) are fulfilled, and $\tau_1 \in [0, \tau_{10})$, then the zero equilibrium $E(0, 0, 0, 0)$ is asymptotically stable when $\tau_2 > \tau_{20}$, when the delay τ_2 decreases, the zero equilibrium $E(0, 0, 0, 0)$ loses its stability and a sequence of Hopf bifurcations occur near the zero equilibrium $E(0, 0, 0, 0)$, i.e., a family of periodic orbits bifurcate from the zero equilibrium $E(0, 0, 0, 0)$. A numerical example verifying our theoretical results is given.

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