Vulnerability of Networks and Existence of Fractional Factor Avoiding Certain Channels and Sites

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Abstract— **The isolated toughness and binding number, as the parameters for measuring stability and vulnerable of networks, have been widely used in computer communication networks. The concept of fractional critical deleted graph is used to measure whether there exists a feasible data packet transmission scheme from site to site when certain intermediate channels and sites are damaged. In this paper, we study the relationship between fractional critical deleted graph and vulnerability parameters. It is highlighted that a graph G is a fractional critical deleted graph if its isolated toughness or binding number meets certain conditions. Furthermore, some conditions we give in the article is sharp in some sense.**

Index Terms— **vulnerable of networks, isolated toughness, binding number, fractional (g, f)-factor, fractional (g, f, n')-critical deleted graph, fractional (g, f, n', m)-critical deleted graph**

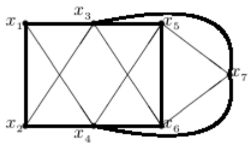

Figure 1. Graph G_1.

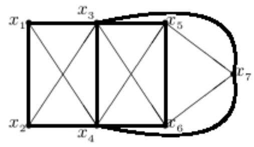

Figure 2. Graph G_2.

I. Introduction

The problem of fractional factor can be considered as a relaxation of the well-known cardinality matching problem. It has wide-ranging applications in areas such as scheduling, network design and the combinatorial polyhedron. For instance, several large data packets to be sent to various destinations through several channels in a communication network. The efficiency of this work can be improved if large data packets to be partitioned into small parcels. The feasible assignment of data packets can be seen as a fractional flow problem and it becomes a fractional factor problem when the destinations and sources of a network are disjoint.

The whole network can be regarded as a graph. Each site correspond to a vertex and each channel correspond to an edge in the graph. Then the model of data transmission problem is just the existence of fractional factor in the graph.

The graphs considered here are finite and simple. Let G be a graph correspond to a certain network with the vertex set $V(G)$ and the edge set $E(G)$. For a vertex $x \in V(G)$, we denote by $d_G(x)$ and $N_G(x)$ the degree and the neighborhood of x in G, respectively. Let $\delta(G)$ denote the minimum degree of G. For any $S \subseteq V(G)$, we write $G[S]$ for the subgraph of G induced by S. Let $i(G-S)$ be the number of isolated vertices in $G-S$. The readers can refer to [1] for standard graph theoretic concepts and terms used but undefined in this paper.

Let g and f be two integer-valued functions on $V(G)$ such that $0 \le g(x) \le f(x)$ for all $x \in V(G)$. A spanning subgraph F of G is called a (g, f)*-factor* if $g(x) \le d_F(x) \le f(x)$ for evert vertex $x \in V(G)$. A *fractional* (g, f)*-factor* is a function h that assigns to each edge of a graph G a number in [0,1] so that for each vertex x we have $g(x) \le \sum_{e \in E(x)} h(e) \le f(x)$. If $g(x) = a$, $f(x) = b$ for all $x \in V(G)$, then a fractional (g, f)-factor is a fractional $[a, b]$-factor. Moreover, if $g(x) = f(x) = k$ ($k \ge 1$ is an integer) for all $x \in V(G)$, then a fractional (g, f)-factor is just a fractional k-factor. Several results of fractional (g, f)-factors due to Zhou [2], [3].

As an example, graph G_1 and G_2 described in Fig 1 and Fig 2. Taking $S = \{x_3, x_4\}$ and $T = V(G) - S$, we can check that both G_1 and G_2 have no fractional 3-factor according to Lemma 2 (showed in next section).

A graph G is called a *fractional* (g, f, n')*-critical graph*

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if after deleting any n' vertices from G, the resulting graph still has a fractional (g, f)-factor. Similarly, a graph G is called a (g, f, n')-*critical graph* if after removing any n' vertices from G, the resulting graph admits a (g, f)-factor. Some sufficient conditions for (a, b, n')-critical graphs can refer [4] and [5].

A graph G is called a *fractional* (g, f, m)-*deleted graph* if after deleting any m edges, the resulting graph still has a (g, f)-factor. Fraction deleted graph and fractional critical graph, as extensions of fractional factor, describe the existence of fractional factor in communication networks when certain channels or certain sites are damaged.

The first author of this paper proposed a new concept to deal with the combination situation when some channels and sites are unavailable in networks. A graph G is called a *fractional* (g, f, n', m)-*critical deleted graph* if after delated any n' vertices from G, the resulting graph is still a fractional (g, f, m)-deleted graph. If $m = 1$, then fractional (g, f, n', m)-critical deleted graph is just fractional (g, f, n')-critical deleted graph.

The concept of isolated toughness $I(G)$ was introduced by Yang et al. [6] as follows. If G is not complete,

$$I(G) = \min\{\frac{|S|}{i(G-S)} : S \subseteq V(G), i(G-S) \geq 2\}.$$

Otherwise, $I(G) = +\infty$. Isolated toughness usually regard as a parameter to measure the strength of the networks, and has widely used in communications networks and ontology semantic structure graph.

In [7], Gao et al. studied a isolated toughness condition for graphs to be fractional (g, f, n')-critical. It is determined that G is a fractional (g, f, n')-critical graph if

$$I(G) \geq \begin{cases} \frac{b^2+bn'-1}{a}, & \text{if } b > a \\ b + n', & \text{if } a = b. \end{cases}$$

And $\delta(G) \geq \frac{bn'}{a} + \frac{(b+1)^2}{4a} + b - 1$, where $1 \leq a \leq b$ and $b \geq 2$. Recently, [8] studied a isolated toughness condition for graphs to be fractional (g, f, n')-critical deleted. It is determined that G is a fractional (g, f, n')-critical deleted graph if

$$I(G) > \begin{cases} \frac{b^2+bn'-1}{a}, & \text{if } b > a \\ b + n', & \text{if } a = b. \end{cases}$$

And $\delta(G) \geq \frac{bn'}{a} + \frac{(b+1)^2}{4a} + b - 1$, where $1 \leq a \leq b$ and $b \geq 2$.

The *binding number* $\text{bind}(G)$ of a graph G is defined as follows:

$$\begin{aligned} \text{bind}(G) &= \min\{\frac{|N_G(X)|}{|X|} | \emptyset \neq X \subseteq V(G), \\ &\quad N_G(X) \neq V(G)\}. \end{aligned}$$

Both isolated toughness and binding number were introduced to measure the vulnerable of networks. Studies have shown that there exists strong link between binding number and existence of fractional factor. Several result can refer [9]–[11].

The contributions of this paper are two-fold. First, we give a isolated toughness bound for fractional (g, f, n')-critical deleted graphs; second, certain binding number conditions are obtained for fractional (g, f, n', m)-critical deleted graphs, and some binding number bounds we give in section III are tight in some sense. The organization of this paper is as follows. The relationship between isolated toughness and fractional (g, f, n')-critical deleted graphs is shown in section II. Also, the isolated toughness condition for fractional (a, b, n')-critical deleted graphs is discussed. In Section III, we describe the conditions for fractional (g, f, n', m)-critical deleted graphs from binding number point. Two examples are manifested to draw out the sharpness of these bounds. Again, the sufficient conditions for fractional (a, b, n', m)-critical deleted graphs are considered.

II. Isolated Toughness Condition for Fractional Critical Deleted Graph

In this section, we extend the result in [8] to the general situation. The main result is the following theorem.

Theorem 1: Let G be a graph, n be a non-negative integer, g, f be two non-negative integer-valued functions on $V(G)$, and $a \leq g(x) \leq f(x) \leq b$ for all $x \in V(G)$, where a, b are two integers with $2 \leq a \leq b$. $\delta(G) \geq \frac{bn'}{g} + \frac{(b+1)^2}{4a} + b$. If G satisfies isolated toughness $I(G) > \frac{b^2+bn'-\Delta}{a}$, where $\Delta = b - a$. Then, G is a fractional (g, f, n')-critical deleted graph.

Let

$$\varepsilon(S,T) = \begin{cases} 2, & T \text{ is not independent set} \\ 1, & T \text{ is an independent set} \\ & \text{and } e_G(T, V(G) \setminus (S \cup T)) \geq 1 \\ 0, & \text{Otherwise.} \end{cases}$$

In order to prove Theorem 1, we depend heavily on the following lemma.

Lemma 2: (Gao [12]) Let G be a graph and let g, f be two non-negative integer-valued functions defined on $V(G)$ satisfying $g(x) \leq f(x)$ for all $x \in V(G)$. Let n be a non-negative integer. Then G is a fractional (g, f, n')-critical deleted graph if and only if

$$\begin{aligned} & f(S) - g(T) + d_{G-S}(T) \\ \geq \quad & \max\{f(U) : U \subseteq S, |U| = n'\} + \varepsilon(S,T) \end{aligned}$$

for any disjoint subsets S and T of $V(G)$ with $|S| \geq n'$.

Lemma 3: (Liu and Zhang [13]) Let G be a graph and let $H = G[T]$ such that $\delta(H) \geq 1$ and $1 \leq d_G(x) \leq k-1$ for every $x \in V(H)$ where $T \subseteq V(G)$ and $k \geq 2$. Let $T_1, \ldots, T_{k-1}$ be a partition of the vertices of H satisfying $d_G(x) = j$ for each $x \in T_j$ where we allow some T_j to be empty. If each component of H has a vertex of degree at most $k - 2$ in G, then H has a maximal independent set I and a covering set $C = V(H) - I$ such that

$$\sum_{j=1}^{k-1} (k-j)c_j \leq \sum_{j=1}^{k-1} (k-2)(k-j)i_j,$$

where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for every $j = 1, \ldots, k-1$.

The lemma below can be deduced from Lemma 2.2 in [13].

Lemma 4: (Liu and Zhang [13]) Let G be a graph and let $H = G[T]$ such that $d_G(x) = k-1$ for every $x \in V(H)$ and no component of H is isomorphic to K_k where $T \subseteq V(G)$ and $k \geq 2$. Then there exists a maximal independent set I and the covering set $C = V(H) - I$ of H satisfying

$$|V(H)| \leq \sum_{i=1}^{k}(k-i+1)|I^{(i)}| - \frac{|I^{(1)}|}{2}$$

and

$$|C| \leq \sum_{i=1}^{k}(k-i)|I^{(i)}| - \frac{|I^{(1)}|}{2},$$

where $I^{(i)} = \{x \in I, d_H(x) = k-i\}$, $1 \leq i \leq k$ and $\sum\limits_{i=1}^{k}|I^{(i)}| = |I|$.

Proof of Theorem 1. If G is complete, then the result immediately follows from $\delta(G) \geq \frac{bn'}{a} + \frac{(b+1)^2}{4a} + b$.

Suppose that there exists non-complete graph G satisfies the conditions of Theorem 1, but is not a fractional (g, f, n')-critical deleted graph. By Lemma 2 and $\varepsilon(S,T) \leq 2$, there exists subsets S and T of $V(G)$ such that

$$\begin{aligned} & a|S| + \sum_{x\in T} d_{G-S}(x) - b|T| \\ \leq\ & f(S) - g(T) + d_{G-S}(T) \leq bn' + 1. \quad (1) \end{aligned}$$

We choose subsets S and T such that $|T|$ is minimum. Obviously, $T \neq \emptyset$ and $d_{G-S}(x) \leq g(x) - 1 \leq b-1$ for each $x \in T$.

Let l be the number of the components of $H' = G[T]$ which are isomorphic to K_b and let $T_0 = \{x \in V(H') | d_{G-S}(x) = 0\}$. Let H be the subgraph obtained from $H' - T_0$ by deleting those l components isomorphic to K_b.

If $|V(H)| = 0$, then $i(G - S \cup S') = |T_0| + l \geq 1$ and we deduce $|S| \leq \frac{b(|T_0|+l)+bn'+1}{a}$ by (1). Let S' be set of vertices such that it contains exactly $b-1$ vertices in each component of K_b in H'. If $i(G - S \cup S') > 1$, then $I(G) \leq \frac{|S\cup S'|}{i(G-S-S')} \leq \frac{b(|T_0|+l)+bn'+al(b-1)+1}{a(|T_0|+l)} \leq \frac{b}{a} + \frac{bn'}{2a} + b - 1 + \frac{1}{2a}$, which contradicts $I(G) > \frac{b^2+bn'-1}{a}$. If $i(G - S \cup S') = 1$, then $d_{G-S}(x) + |S| \geq d_G(x) \geq \delta(G) \geq \frac{bn'}{a} + \frac{b}{a} + b$ and $d_{G-S}(x) \geq \frac{bn'}{a} + \frac{(b+1)^2}{4a} + b - |S| \geq \frac{bn'}{a} + \frac{b}{a} + b - \frac{b(n'+1)+1}{a}$, which contradicts $d_{G-S}(x) \leq b-1$ for any $x \in T$.

By $|V(H)| \geq 1$, we assume $H = H_1 \cup H_2$ where H_1 is the union of components of H which satisfies that $d_{G-S}(x) = b-1$ for every vertex $x \in V(H_1)$ and $H_2 = H - H_1$. From Lemma 4, there exists a maximum independent set I_1 and the covering set $C_1 = V(H_1) - I_1$ of H_1 such that

$$|V(H_1)| \leq \sum_{i=1}^{b}(b-i+1)|I^{(i)}| - \frac{|I^{(1)}|}{2} \quad (2)$$

and

$$|C_1| \leq \sum_{i=1}^{b}(b-i)|I^{(i)}| - \frac{|I^{(1)}|}{2}, \quad (3)$$

where $I^{(i)} = \{x \in I_1, d_{H_1}(x) = b-i\}$, $1 \leq i \leq b$ and $\sum\limits_{i=1}^{b}|I^{(i)}| = |I_1|$. Let $T_j = \{x \in V(H_2) | d_{G-S}(x) = j\}$ for $1 \leq j \leq b-1$. Each component of H_2 has a vertex of degree at most $b-2$ in $G-S$ by the definitions of H and H_2. According to Lemma 3, H_2 has a maximal independent set I_2 and the covering set $C_2 = V(H_2) - I_2$ such that

$$\sum_{j=1}^{b-1}(b-j)c_j \leq \sum_{j=1}^{b-1}(b-2)(b-j)i_j, \quad (4)$$

where $c_j = |C_2 \cap T_j|$ and $i_j = |I_2 \cap T_j|$ for every $j = 1, \ldots, b-1$. Set $W = V(G) - S - T$ and $U = S \cup S' \cup C_1 \cup (N_G(I_1) \cap W)) \cup C_2 \cup (N_G(I_2) \cap W)$. We get

$$|U| \leq |S| + l(b-1) + |C_1| + \sum_{j=1}^{b-1} j i_j + \sum_{i=1}^{b}(i-1)|I^{(i)}| \quad (5)$$

and

$$i(G-U) \geq t_0 + l + |I_1| + \sum_{j=1}^{b-1} i_j, \quad (6)$$

where $t_0 = |T_0|$. Then when $i(G-U) > 1$, we infer

$$|U| \geq I(G) i(G-U). \quad (7)$$

If $i(G-U) = 1$, we yield

$$\begin{aligned} |S| &\leq \frac{bn' + b|T| - d_{G-S}(T) + 1}{a} \\ &\leq \frac{bn' + b|T| - |T|(|T|-1) + 1}{a} \\ &\leq \frac{bn' + 1 + b\frac{b+1}{2} - (\frac{b+1}{2})(\frac{b+1}{2} - 1)}{a} \\ &= \frac{bn'+1}{a} + \frac{(b+1)^2}{4a}, \end{aligned}$$

and $d_{G-S}(x) \geq \frac{bn'}{a} + \frac{(b+1)^2}{4a} + b - |S| \geq \frac{bn'}{a} + \frac{(b+1)^2}{4a} + b - (\frac{bn'+1}{a} + \frac{(b+1)^2}{4a})$, which contradicts $d_{G-S}(x) \leq b-1$ for each $x \in T$.

In terms of (5), (6) and (7), we have

$$\begin{aligned} & |S| + |C_1| \quad (8) \\ \geq\ & \sum_{j=1}^{b-1}(I(G) - j)i_j + I(G)(t_0 + l) + I(G)|I_1| \\ & - \sum_{i=1}^{b}(i-1)|I^{(i)}| - l(b-1). \quad (9) \end{aligned}$$

By $b|T| - d_{G-S}(T) \geq a|S| - bn' - 1$, we obtain

$$\begin{aligned} & bt_0 + bl + |V(H_1)| + \sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j \\ \geq\ & a|S| - bn' - 1. \end{aligned}$$

Thus, we obtain

$$\begin{aligned}&|V(H_1)| + \sum_{j=1}^{b-1}(b-j)c_j + a|C_1|\\ \geq\ & \sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j + (aI(G)-b)(t_0+l)\\ &+aI(G)|I_1| - a\sum_{i=1}^{b}(i-1)|I^{(i)}|\\ &-bn'-1-la(b-1). \qquad (10)\end{aligned}$$

In view of (2) and (3), we deduce

$$\begin{aligned}&|V(H_1)| + a|C_1|\\ \leq\ & \sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}| \qquad (11)\\ &-\frac{(a+1)|I^{(1)}|}{2}. \qquad (12)\end{aligned}$$

Combining (4), (10) and (11), we get

$$\begin{aligned}&\sum_{j=1}^{b-1}(b-2)(b-j)i_j + \sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}|\\ \geq\ & \sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j + aI(G)|I_1|\\ &+\frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}|\\ &+(aI(G)-b)(t_0+l)-bn'-la(b-1)-1. \qquad (13)\end{aligned}$$

The following argument separates into two cases according to the value of t_0+l.

Case 1. $t_0+l\geq 1$**.** By $aI(G) > b^2+bn'-\Delta$, we have $(aI(G)-b)(t_0+l)-bn'-la(b-1)-1\geq 0$ by $\Delta\geq 0$ and $a\geq 2$. Therefore, (13) becomes

$$\begin{aligned}&\sum_{j=1}^{b-1}(b-2)(b-j)i_j + \sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}|\\ \geq\ & \sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j + aI(G)|I_1|\\ &+\frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}|.\end{aligned}$$

Hence, at least one of the following two cases must hold.

Subcase 1. There is at least one j such that

$$(b-2)(b-j)\geq aI(G)-aj-b+j,$$

which implies

$$\begin{aligned}aI(G) &< b(b-2)+(a-b+1)j+b\\ &\leq b(b-2)+(a-b+1)+b\\ &= (b^2-1)+(a-b)+(2-b)\\ &< b^2-\Delta,\end{aligned}$$

which contradicts $I(G) > \frac{b^2-\Delta+bn'}{a}$.

Subcase 2.

$$\begin{aligned}&\sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}|\\ \geq\ & aI(G)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}|\\ \geq\ & (b^2+bn'-\Delta)|I_1| + \frac{(a+1)|I^{(1)}|}{2}\\ &-a\sum_{i=1}^{b}(i-1)|I^{(i)}|\\ \geq\ & (b^2-\Delta)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}|.\end{aligned}$$

This implies,

$$\begin{aligned}&\sum_{i=2}^{b}(ab+2b-2a-i+1-b^2+\Delta)|I^{(i)}|\\ &+(ab+2b-\frac{5}{2}a-b^2-\frac{1}{2})|I^{(1)}|\geq 0.\end{aligned}$$

Let

$$h_1(b) = -b^2+(a+2)b-\frac{5}{2}a-\frac{1}{2}.$$

From $b\geq a$, we get

$$\max\{h_1(b)\} = h_1(a) < 0.$$

On the other hand, $ab+2b-2a-i+1-b^2\leq -b^2+(a+2)b-2a-1$ due to $i\geq 2$. Let

$$h_2(b) = -b^2+(a+2)b-2a-1.$$

We infer

$$\max\{h_2(b)\} = h_2(a) < 0$$

by $b\geq a$. This is a contradiction.

Case 2. $t_0+l=0$**.** In this case, (13) becomes

$$\begin{aligned}&\sum_{j=1}^{b-1}(b-2)(b-j)i_j + \sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}|\\ \geq\ & \sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j + aI(G)|I_1|\\ &+\frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}| - bn'-1.\end{aligned}$$

From what we have discussed in Subcase 1, we get $\sum_{j=1}^{b-1}(b-2)(b-j)i_j \leq \sum_{j=1}^{b-1}(at-aj-b+j)i_j - 1$. If

$|I_1| > 0$, we deduce

$$\begin{aligned}
&\sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}| \\
\geq\ & aI(G)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}| \\
&-bn' \\
\geq\ & (b^2+bn-\Delta)|I_1| + \frac{(a+1)|I^{(1)}|}{2} \\
&-a\sum_{i=1}^{b}(i-1)|I^{(i)}| - bn' \\
\geq\ & (b^2-\Delta)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}|.
\end{aligned}$$

The result follows from what we discussed in Subcase 2 above.

The last situation is $|I_1| = 0$ and $\sum_{j=1}^{b-1}(b-2)(b-j)i_j \geq \sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j - bn' - 1$. Let $h_3 = (b-2)(b-j) - (aI(G)-aj-b+j) + bn' + 1$. By $b \geq a$ and $a \geq 2$, we infer

$$\begin{aligned}
h_3 &= b(b-2) + (a-b+1)j + b - aI(G) + bn' + 1 \\
&< b(b-2) + (a-b+1) + b - (b^2+bn'-\Delta) \\
&\quad +bn' + 1 \\
&= -b+2 \leq 0,
\end{aligned}$$

a contradiction.

From what we argument above, we deduce the desired contradictions, and hence we can conclude that Theorem 1 holds. □

Remark 1. (Tight isolated toughness condition for fractional (a,b,n)-critical graph) Let $g(x) = a$, $f(x) = b$ for each $x \in V(G)$. The sufficient and necessity condition for fractional (a,b,n')-critical deleted graph derives from Lemma 1.

Lemma 5: Let G be a graph. Let a, b, n' be non-negative integers such that $a \leq b$. Then G is a fractional (a,b,n')-critical deleted graph if and only if

$$b|S| - a|T| + d_{G-S}(T) \geq bn' + \varepsilon(S,T) \qquad (14)$$

for all disjoint subsets S, T of $V(G)$ with $|S| \geq n'$.

Using standard techniques similar to that of Section II. We get following result. We skip the detail proof.

Theorem 6: Let G be a graph and let a, b be two nonnegative integers satisfying $2 \leq a \leq b$. Let n' be a non-negative integer. $|V(G)| \geq n'+a+1$ if G is complete. If $I(G) > \frac{ab-b+a-\Delta}{b} + n'$, then G is a fractional (a,b,n')-critical deleted graph.

III. Binding Number Condition for Fractional Critical Deleted Graphs

A. Main Results in This Section

In [14], Gao and Wang gave following result on fractional (g,f,n',m)-critical deleted graphs.

Theorem 7: [14] Let G be a graph of order n, and let a, b, n', and m be non-negative integers such that $2 \leq a \leq b$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If $\mathrm{bind}(G) > \frac{(a+b-1)(n-1)}{an-(a+b)-bn'-2m+2}$ and $n \geq \frac{(a+b)(a+b-3)}{a} + \frac{bn'+2m}{a-1}$, then G is a fractional (g,f,n',m)-critical deleted graph.

Theorem 8: [14] Let G be a graph of order n, and let a, b, n', and m be non-negative integers such that $2 \leq a \leq b$, $n \geq \frac{(a+b-1)(a+b-2)-2}{a} + \frac{bn'+2m}{a-1}$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If G satisfies

$$\mathrm{bind}(G) \geq \frac{(a+b-1)(n-1)}{a(n-1)-bn'-2m}$$

and

$$\delta(G) \neq \lfloor \frac{(b-1)n+a+b+bn'+2m-2}{a+b-1} \rfloor,$$

then G is a fractional (g,f,n',m)-critical deleted graph.

Also, above two results are sharp if $a = b$. The aim of this section is to strengthen these results when a and b are both even integers. The main results in this section are stated as follows.

Theorem 9: Let G be a graph of order n, and let a, b, n', and m be non-negative integers such that such that a and b are two even integers with $2 \leq a \leq b$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If $\mathrm{bind}(G) > \frac{(a+b-1)(n-1)}{an-(a+b)-bn'-2m+3}$ and $n \geq \frac{(a+b)(a+b-3)}{a} + \frac{bn'+2m}{a-1}$, then G is a fractional (g,f,n',m)-critical deleted graph.

Theorem 10: Let G be a graph of order n, and let a, b, n', and m be non-negative integers such that such that a and b are two even integers with $2 \leq a \leq b$, $n \geq \frac{(a+b-1)(a+b-2)-2}{a} + \frac{bn'+2m}{a-1}$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If G satisfies

$$\mathrm{bind}(G) \geq \frac{(a+b-1)(n-1)}{a(n-1)-bn'-2m+1}$$

and

$$\delta(G) \neq \lfloor \frac{(b-1)n+a+b+bn'+2m-3}{a+b-1} \rfloor,$$

then G is a fractional (g,f,n',m)-critical deleted graph.

The proofs of Theorem 9 and Theorem 10 are based on the following lemma:

Lemma 11: (Gao [12]) Let G be a graph, g, f be two integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$. Let n', m be two non-negative integers. Then G is fractional (g,f,n',m)-critical deleted graph if and only if

$$\begin{aligned}
&f(S) - g(T) + d_{G-S}(T) \\
\geq\ & \max_{U\subseteq S, |U|=n', H\subseteq E(G-U), |H|=m}\{f(U) \\
&+ \sum_{x\in T} d_H(x) - e_H(T,S)\} \qquad (15)
\end{aligned}$$

for all disjoint subsets S, T of $V(G)$ with $|S| \geq n'$.

B. Proof of Theorem 9

Suppose that G satisfies conditions of Theorem 9 but is not a fractional (g,f,n',m)-critical deleted graph. Obviously, $T \neq \emptyset$. Otherwise, (15) holds. By Lemma 11 and the fact $\sum_{x\in T} d_H(x) - e_H(T,S) \leq 2m$, there exists disjoint subsets S and T of $V(G)$ such that

$$f(S) - g(T) + d_{G-S}(T) \leq bn' + 2m - 1, \qquad (16)$$

where $|S| \geq n'$. We choose S and T such that $|T|$ is minimum. Thus, for each $x \in T$, we have $d_{G-S}(x) \leq g(x) - 1 \leq b-1$. Otherwise, if there exists some $x \in T$ such that $d_{G-S}(x) \geq g(x)$, then S and $T\backslash\{x\}$ also satisfy (16). This contradicts the choice of S and T.

Let $d = \min\{d_{G-S}(x) : x \in T\}$. Then $0 \leq d \leq b-1$, and

$$f(S) + d_{G-S}(T) - f(T) \geq a|S| + d|T| - b|T|.$$

Thus,

$$bn' + 2m - 1 \geq a|S| - (b-d)|T|. \qquad (17)$$

We choose $x_1 \in T$ such that $d_{G-S}(x_1) = d$. We shall get some contradictions in the following two cases.

Case 1. $1 \leq d \leq b-1$.

Let $Y = (V(G)\setminus S)\setminus N_{G-S}(x_1)$. Then $x_1 \in Y \setminus N_G(Y)$. Thus, $Y \neq \emptyset$, $N_G(Y) \neq V(G)$, and $|N_G(Y)| \geq \mathrm{bind}(G)|Y|$. We get

$$\begin{aligned} & n-1 \geq |N_G(Y)| \\ \geq\ & \mathrm{bind}(G)|Y| = \mathrm{bind}(G)(n-d-|S|), \end{aligned}$$

that is to say,

$$\begin{aligned} & |S| \\ \geq\ & n - d - \frac{n-1}{\mathrm{bind}(G)} \\ >\ & n - d - \frac{an-(a+b)-bn'-2m+2}{a+b-1} \\ =\ & \frac{(b-d)n+(a+b)+bn'+2m-3}{a+b-1} - d. \qquad (18) \end{aligned}$$

Since a and b are even integers, we obtain

$$|S| \geq \frac{(b-d)n+bn+2m-2}{a+b-d}. \qquad (19)$$

By (19), we get

$$\begin{aligned} & \frac{(b-d)n+bn+2m-2}{a+b-d} \\ >\ & \frac{(b-1)n+a+b+bn'-3}{a+b-1} - d \qquad (20) \end{aligned}$$

If $d = 1$, then we infer $\frac{(b-1)n+bn+2m-2}{a+b-1} > \frac{(b-1)n+a+b+bn'-3}{a+b-1} - 1$ by (20). A contradiction.

We denote the left-hand and right-hand side of (20) as A and B respectively, then we have $A - B > 0$. We multiply $(a+b-1)(a+b-d)$ and rearrange, thus get

$$\begin{aligned} & 0 \\ <\ & (a+b-1)(a+b-d)(A-B) \\ =\ & (a+b-1)(a+b-d)(\frac{(b-d)n+bn+2m-2}{a+b-d} \\ & -\frac{(b-1)n+a+b+bn'-3}{a+b-1} + d) \\ =\ & -(d-1)(an-(a+b-1)(a+b-d) \\ & -bn'-2m+2) \end{aligned}$$

Combining with $2 \leq d \leq b-1$, we deduce

$$n < \frac{(a+b-1)(a+b-d)+bn'+2m-2}{a},$$

which contradicts that $n \geq \frac{(a+b)(a+b-3)}{a} + \frac{bn'+2m}{a-1}$.

Case 2. $d = 0$.

In this case, we first show the following claim.

Claim 1: $\frac{an-(a+b)-bn'-2m+3}{n-1} \geq 1$.

Proof of Claim 1. Since $n \geq \frac{(a+b)(a+b-3)}{a} + \frac{bn'+2m}{a-1}$, we have

$$\begin{aligned} & an-(a+b)-bn'-2m+3-(n-1) \\ =\ & (a-1)n-(a+b)-bn'-2m+4 \\ \geq\ & (a-1)(\frac{(a+b)(a+b-3)}{a} + \frac{bn'+2m}{a-1}) \\ & -(a+b)-bn'-2m+4 \\ =\ & \frac{(a-1)(a+b)(a+b-3)}{a} - (a+b) + 4 \\ \geq\ & 2(a+b-3)-(a+b)+4 \\ =\ & a+b-4 \geq 0. \end{aligned}$$

Thus, we get $\frac{an-(a+b)-bn'-2m+2}{n-1} > 1$. □

Let $h = |\{x : x \in T, d_{G-S}(x) = 0\}|$, and $Y = V(G)\setminus S$. By $d = 0$, we have $N_G(Y) \neq V(G)$. Also, by $T \neq \emptyset$, we have $Y \neq \emptyset$. So, $|N_G(Y)| \geq \mathrm{bind}(G)|Y|$. Therefore,

$$n - h \geq |N_G(Y)| \geq \mathrm{bind}(G)|Y| = \mathrm{bind}(G)(n-|S|).$$

So,

$$\begin{aligned} & |S| \geq n - \frac{n-h}{\mathrm{bind}(G)} \\ >\ & n - \frac{(n-h)(an-(a+b)-bn'-2m+3)}{(a+b-1)(n-1)} \qquad (21) \end{aligned}$$

By (17), (21), Claim 2 and the fact $|T| \leq n - |S|$, we get

$$\begin{aligned}
& bn' + 2m - 1 \\
\geq\ & f(S) + d_{G-S}(T) - g(T) \\
\geq\ & a|S| + |T| - h - b|T| = a|S| - (b-1)|T| - h \\
\geq\ & a|S| - (b-1)(n - |S|) - h = (a+b-1)|S| \\
& -(b-1)n - h \\
>\ & (a+b-1)(n \\
& - \frac{(n-h)(an-(a+b)-bn'-2m+3)}{(a+b-1)(n-1)}) \\
& -(b-1)n - h \\
=\ & an - \frac{(n-h)(an-(a+b)-bn'-2m+3)}{n-1} - h \\
\geq\ & an - \frac{(n-1)(an-(a+b)-bn'-2m+3)}{n-1} - 1 \\
=\ & bn' + 2m + (a+b) - 4 \geq bn' + 2m,
\end{aligned}$$

which is a contradiction. □

C. Sharpness of Theorem 9

The condition that $\mathrm{bind}(G) > \frac{(a+b-1)(n-1)}{(an-(a+b)-bn'-2m+3)}$ in Theorem 9 cannot be replaced by $\mathrm{bind}(G) \geq \frac{(a+b-1)(n-1)}{(an-(a+b)-bn'-2m+3)}$. Let $2 \leq a = b$ be two even integers, and $n', m \geq 0$ be integers. Let $n = ((a+b-1)(a+b-2)+(a+b-3)+(a+2b-1)n'+(a+b+1)m)/a$, $l = (a+b+n'+m-1)/2$, and $h = n - 2l = n - (a+b+n'+m-1) = ((a+b-1)(b-2) + (a+b-3) + (2b-1)n' + (b+1)m)/a$ be integers. Let $G = K_h \vee lK_2$. $X = V(lK_2)$, and for each $x \in X$, we have $|N_H(X \setminus x)| = n - 1$. By the definition of bind(G), we have

$$\begin{aligned}
\mathrm{bind}(G) &= \frac{N_G(X \setminus x)}{X \setminus x} = \frac{n-1}{2l-1} \\
&= \frac{(a+b-1)(n-1)}{an-(a+b)-bn'-2m+3}.
\end{aligned}$$

Let $S = V(K_h)$, $T = V(lK_2)$, H be any subgraph of $G[T]$ with m edges. Then $|S| = h \geq n'$, $|T| = 2l$, and $\sum_{x \in T} d_H(x) - e_H(S,T) = 2m$. Since $a = b$, we have $g(x) = a = b = f(x)$ for all $x \in V(H)$. Thus,

$$\begin{aligned}
& f(S) + d_{G-S}(T) - g(T) \\
=\ & a|S| - (b-1)|T| \\
=\ & a\{\frac{(a+b-1)(b-2) + (a+b-3)}{a} \\
& + \frac{(2b-1)n' + (b+1)m}{a}\} \\
& -(b-1)(a+b+n'+m-1) \\
=\ & bn' + 2m - 2 < bn' + 2m \\
=\ & \max_{U \subseteq S, |U| = n', H \subseteq E(G-U), |H| = m} \{f(U) \\
& + \sum_{x \in T} d_H(x) - e_H(T,S)\}.
\end{aligned}$$

By Lemma 11, G is not a fractional (g,f,n',m)-critical deleted graph. Thus, the binding number condition in Theorem 9 is tight.

D. Proof of Lemma 12

To show Theorem 10, we need the following lemma, which is a neighborhood condition for a graph G to be a fractional (g,f,n',m)-critical deleted graph.

Lemma 12: Let a, b, n', n and m are non-negative integers such that a and b are two even integers with $2 \leq a \leq b$. Let G be a graph with order n such that $n \geq \frac{(a+b-1)(a+b-2)+bn'+2m-2}{a}$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If

$$|N_G(X)| > \frac{(b-1)n + |X| + bn' + 2m - 2}{a+b-1}$$

holds for each non-empty independent subset $X \subseteq V(G)$, and

$$\delta(G) > \frac{(b-1)n + a + b + bn' + 2m - 3}{a+b-1},$$

then G is a fractional (g,f,n',m)-critical deleted graph.

Suppose that G satisfies conditions of Lemma 12, but is not a fractional (g,f,n',m)-critical deleted graph. Obviously, $T \neq \emptyset$. By Lemma 11, there exists disjoint S and T satisfying

$$f(S) - g(T) + d_{G-S}(T) \leq bn' + 2m - 1, \tag{22}$$

where $|S| \geq n'$. We choose S and T such that $|T|$ is minimum. Thus, we have $d_{G-S}(x) \leq g(x) - 1 \leq b - 1$ for each $x \in T$.

Let $d = \min\{d_{G-S}(x) | x \in T\}$, then

$$0 \leq d \leq b - 1,$$

$$\delta(G) \leq d + |S|. \tag{23}$$

In terms of (23) and a, b are even integers, we get

$$|S| \geq \delta(G) - d > \frac{(b-1)n + a + b + bn' - 3}{a+b-1} - d. \tag{24}$$

Now, we consider the following two cases according to the value of d.

Case 1. $1 \leq d \leq b - 1$.

Since a and b are even integers, we obtain

$$|S| \geq \frac{(b-d)n + bn + 2m - 2}{a+b-d}. \tag{25}$$

By (24) and (25), we get

$$\frac{(b-d)n + bn + 2m - 2}{a+b-d} > \frac{(b-1)n + a + b + bn' - 3}{a+b-1} - d \tag{26}$$

If $d = 1$, then we infer $\frac{(b-1)n+bn+2m-2}{a+b-1} > \frac{(b-1)n+a+b+bn'-3}{a+b-1} - 1$ by (26). A contradiction.

We denote the left-hand and right-hand side of (26) as A and B respectively, then we have $A - B > 0$. We multiply $(a+b-1)(a+b-d)$ and rearrange, thus get

$$\begin{aligned}
& 0 \\
< \;& (a+b-1)(a+b-d)(A-B) \\
= \;& (a+b-1)(a+b-d)(\frac{(b-d)n+bn+2m-2}{a+b-d} \\
& -\frac{(b-1)n+a+b+bn'-3}{a+b-1}+d) \\
= \;& -(d-1)(an-(a+b-1)(a+b-d) \\
& -bn'-2m+2)
\end{aligned}$$

Combining with $2 \le d \le b-1$, we deduce

$$\begin{aligned}
n \;&<\; \frac{(a+b-1)(a+b-d)+bn'+2m-2}{a} \\
&\le\; \frac{(a+b-1)(a+b-2)+bn'+2m-2}{a},
\end{aligned}$$

which contradicts that $n \ge \frac{(a+b-1)(a+b-2)+bn'+2m-2}{a}$.

Case 2. $d=0$.

Let $Y = \{x \in T \mid d_{G-S}(x) = 0\}$. Obviously, $Y \neq \emptyset$, and Y is an independent set. Thus, by Lemma 12, we have

$$\begin{aligned}
& \frac{(b-1)n+|Y|+bn'+2m-2}{a+b-1} \\
< \;& |N_G(Y)| \le |S|. \qquad (27)
\end{aligned}$$

Subcase 2.1 $|S|+|T| \le n-1$.

In view of (22) and $|S|+|T| \le n$, we get

$$\begin{aligned}
& bn'+2m-1 \\
\ge \;& f(S)+d_{G-S}(T)-g(T) \\
\ge \;& a|S|+d_{G-S}(T)-b|T| \\
\ge \;& a|S|+|T|-|Y|-b|T| \\
= \;& a|S|-(b-1)|T|-|Y| \\
\ge \;& a|S|-(b-1)(n-1-|S|)-|Y| \\
= \;& (a+b-1)|S|-(b-1)n-|Y|+b-1 \\
\ge \;& (a+b-1)|S|-(b-1)n-|Y|+1,
\end{aligned}$$

which implies

$$|S| \le \frac{(b-1)n+|Y|+bn'+2m-2}{a+b+1},$$

which contradicts 27.

Subcase 2.2 $|S|+|T| = n$.

From (22) and (24), we have

$$\begin{aligned}
& bn'+2m-1 \\
\ge \;& f(S)+d_{G-S}(T)-g(T) \\
\ge \;& a|S|+d_{G-S}(T)-b|T| \\
\ge \;& a|S|+|T|-|Y|-b|T| \\
= \;& a|S|-(b-1)|T|-|Y| \\
\ge \;& a|S|-(b-1)(n-|S|)-|Y| \\
= \;& (a+b-1)|S|-(b-1)n-|Y| \\
\ge \;& (a+b-1)(\frac{(b-1)n+|Y|+bn'+2m-1}{a+b-1}) \\
& -(b-1)n-|Y| \\
= \;& bn'+2m-1.
\end{aligned}$$

This implies

$$d_{G-S}(T) = |T|-|Y| \qquad (28)$$

and

$$a|S|+d_{G-S}(T)-b|T| = bn'+2m-1 \qquad (29)$$

We get $d_{G-S}(T)$ is even. To see this, we observe that $Y \subseteq T$. If $|T|=|Y|$, then $d_{G-S}(T)=0$ from (28). If $|T|>|Y|$, we have $d_{G-S}(v)=1$ for each $v \in T-Y$ by (28) and definition of Y. Since $|S|+|T|=n$, we get $d_{G[T-Y]}(v)=1$ for each $v \in T-Y$, and $G[T-Y]$ is a perfect matching. Thus, $|T|-|Y|$ is even, and so is $d_{G-S}(T)$.

By a and b are even, we get $a|S|+d_{G-S}(T)-b|T|$ is even, which contradicts (29). □

E. Sharpness of Lemma 12

Let $b = a \ge 2$ be two even integers that $\frac{(a+b-1)(a+b-2)}{2(b-1)}$ is an integer. We write $n = \frac{(a+b-1)(a+b-2)}{b-1} + 4m + n'$. The following example shows that the neighborhood condition $|N_G(X)| > \frac{(b-1)n+|X|+bn'+2m-2}{a+b-1}$ in Lemma 12 cannot be replaced by $|N_G(X)| \ge \frac{(b-1)n+|X|+bn'+2m-1}{a+b-2}$. Let $G = K_{a+b+n'+2m-1} \vee ((a+b+1)K_1 \cup (\frac{(a+b-1)(a+b-2)}{2(b-1)} + m-(a+b))K_2)$ and $Y = V((a+b+1)K_1)$. Then $\delta(G) = a+b+n'+2m-1 > \frac{(b-1)n+a+b+bn'+2m-3}{a+b-1}$ and $|N_G(Y)| = a+b+n'+2m-1 = \frac{(b-1)n+|Y|+bn'+2m-2}{a+b-1}$. For each non-empty independent subset X of $V(G)$, we infer $|N_G(X)| \ge \frac{(b-1)n+|X|+bn'+2m-2}{a+b-1}$. Let $S = V(K_{a+b+n'+2m-1})$ and $T = V((a+b+1)K_1 \cup (\frac{(a+b-1)(a+b-2)}{2(b-1)}+m-(a+b))K_2)$. Then $|S| = a+b+n'+2m-1$, $|T| = \frac{(a+b-1)(a+b-2)}{b-1}-(a+b)+2m+1$, and $d_{G-S}(T) = \frac{(a+b-1)(a+b-2)}{b-1}-2(a+b)+2m$. Since $a=b$, we have $f(v)=g(v)=a=b$ for each $v \in V(G)$. Thus, we have

$$\begin{aligned}
& f(S)+d_{G-S}(T)-g(T) \\
= \;& b(a+b+n'+2m-1)+\frac{(a+b-1)(a+b-2)}{b-1} \\
& -2(a+b)+2m-a(\frac{(a+b-1)(a+b-2)}{b-1} \\
& -(a+b)+2m+1) \\
= \;& bn'+2m-2 \\
< \;& \max_{U\subseteq S,|U|=n',H\subseteq E(G-U),|H|=m}\{f(U) \\
& +\sum_{x\in T} d_H(x)-e_H(T,S)\}.
\end{aligned}$$

Namely, G is not a fractional (g,f,n',m)-critical deleted graph. In this sense, the condition of $|N_G(X)|$ in Lemma 12 is best possible.

F. Proof of Theorem 10.

Now, we begin to prove Theorem 10.

Suppose that G satisfies the conditions of Theorem 10, but is not a fractional (g,f,n',m)-critical deleted graph.

Obviously, $T \neq \emptyset$. By Lemma 11, there exists disjoint subsets S and T satisfying

$$f(S) - g(T) + d_{G-S}(T) \leq bn' + 2m - 1, \quad (30)$$

where $|S| \geq n'$. We choose S and T such that $|T|$ is minimum. Then $d_{G-S}(x) \leq g(x) - 1 \leq b - 1$ for each $x \in T$.

For each $X \subseteq V(G)$, $X \neq \emptyset$ and $N_G(X) \neq V(G)$. Let $Y = V(G) \setminus N_G(X)$. Clearly, $\emptyset \neq Y \subseteq V(G)$.

Claim 2: $X \cap N_G(Y) = \emptyset$.

Proof of Claim 2. Assume that $X \cap N_G(Y) \neq \emptyset$, say $x \in X \cap N_G(Y)$. By $x \in N_G(Y)$, we have $y \in Y$ and $xy \in E(G)$. Thus, $y \in N_G(x) \subseteq N_G(X)$, contradicting $y \in Y = V(G) \setminus N_G(X)$. □

Claim 3: $|N_G(X)| > \frac{(b-1)n+|X|+bn'+2m-1}{a+b-1}$.

Proof of Claim 3. Using Claim 2, we have

$$|X| + |N_G(Y)| \leq n \quad (31)$$

and

$$N_G(Y) \neq V(G). \quad (32)$$

According to (31), (32) and the definition of bind(G), we get

$$\begin{aligned} \text{bind}(G) &\leq \frac{|N_G(Y)|}{|Y|} \\ &\leq \frac{n - |X|}{|V(G) \setminus N_G(X)|} \\ &= \frac{n - |X|}{n - |N_G(X)|}. \end{aligned} \quad (33)$$

From (33), we have

$$|N_G(X)| \geq n - \frac{n - |X|}{\text{bind}(G)}. \quad (34)$$

Let $F(t) = n - \frac{n-|X|}{t}$. Then, by $X \subseteq V(G)$, we obtain

$$F'(t) = \frac{n - |X|}{t^2} \geq 0.$$

Combining this with $\text{bind}(G) \geq \frac{(a+b-1)(n-1)}{a(n-1)-bn'-2m+1}$, we get

$$F(\text{bind}(G)) \geq F(\frac{(a+b-1)(n-1)}{a(n-1) - bn' - 2m + 1}).$$

Thus,

$$\begin{aligned} & n - \frac{n - |X|}{\text{bind}(G)} \\ \geq\ & n - \frac{n - |X|}{\frac{(a+b-1)(n-1)}{a(n-1)-bn'-2m+1}} \\ =\ & n - \frac{(n - |X|)(a(n-1) - bn' - 2m + 1)}{(a + b - 1)(n - 1)} \end{aligned} \quad (35)$$

By (34), (35) and $n \geq \frac{(a+b-1)(a+b-2)-2}{a} + \frac{bn'+2m-1}{a-1}$, we obtain

$$\begin{aligned} & |N_G(X)| \\ \geq\ & n - \frac{n - |X|}{bind(G)} \\ \geq\ & n - \frac{(n - |X|)(a(n-1) - bn' - 2m + 1)}{(a + b - 1)(n - 1)} \\ =\ & \frac{(b-1)(n-1)n + (a(n-1) - bn' - 2m + 1)|X|}{(a + b - 1)(n - 1)} \\ & + \frac{(bn' + 2m - 1)n}{(a + b - 1)(n - 1)} \\ =\ & \frac{(b-1)(n-1)n + (n-1)|X|}{(a + b - 1)(n - 1)} \\ & + \frac{((a-1)(n-1) - bn' - 2m + 1)|X|}{(a + b - 1)(n - 1)} \\ & + \frac{(bn' + 2m - 1)n}{(a + b - 1)(n - 1)} \\ \geq\ & \frac{(b-1)(n-1)n + (n-1)|X|}{(a + b - 1)(n - 1)} \\ & + \frac{((a-1)(n-1) - bn' - 2m + 1)}{(a + b - 1)(n - 1)} \\ & + \frac{(bn' + 2m - 1)n}{(a + b - 1)(n - 1)} \\ =\ & \frac{(b-1)(n-1)n + (n-1)|X|}{(a + b - 1)(n - 1)} \\ & \frac{(a-1)(n-1) + (bn' + 2m - 1)(n - 1)}{(a + b - 1)(n - 1)} \\ =\ & \frac{(b-1)n + |X| + bn' + 2m - 1 + a - 1}{a + b - 1} \\ >\ & \frac{(b-1)n + |X| + bn' + 2m - 2}{a + b - 1} \end{aligned}$$

Therefore, Claim 4 holds. □

Since each $\emptyset \neq X \subseteq V(G)$ satisfies $|N_G(X)| \geq \frac{(b-1)n+|X|+bn'+2m+a-2}{a+b-1}$, we get

$$\delta(G) \geq \frac{(b-1)n + a + bn' + 2m - 1}{a + b - 1}. \quad (36)$$

Claim 4: $\delta(G) > \frac{(b-1)n+a+b+bn'+2m-3}{a+b-1}$.

Proof of Claim 4. Suppose that $\delta(G) \leq \frac{(b-1)n+a+b+bn'+2m-3}{a+b-1}$. By (36),

$$\begin{aligned} & \lceil \frac{(b-1)n + a + bn' + 2m - 1}{a + b - 1} \rceil \\ \leq\ & \delta(G) \\ \leq\ & \lfloor \frac{(b-1)n + a + b + bn' + 2m - 3}{a + b - 1} \rfloor. \end{aligned}$$

That is,

$$\begin{aligned} & \lceil \frac{(b-1)n + a + bn' + 2m - 1}{a + b - 1} \rceil \\ =\ & \delta(G) \\ =\ & \lfloor \frac{(b-1)n + a + b + bn' + 2m - 3}{a + b - 1} \rfloor. \end{aligned}$$

This contradicts the condition of Theorem 10. □

Now, the result follows from Claim 3, Claim 4 and Lemma 12. □

G. Some Results on Fractional (a, b, n', m)-critical Deleted Graph

Let $g(x) = a$, $f(x) = b$ for each $x \in V(G)$. The sufficient and necessity condition for fractional (a, b, n', m)-critical deleted graph derives from Lemma 1.

Lemma 13: Let G be a graph. Let a, b, n', m be non-negative integers such that $a \leq b$. Then G is a fractional (a, b, n', m)-critical deleted graph if and only if

$$b|S| - a|T| + d_{G-S}(T) \geq bn' + (\sum_{x\in T} d_H(x) - e_H(T, S)) \tag{37}$$

for all disjoint subsets S, T of $V(G)$ with $|S| \geq n'$.

Using standard techniques similar to that of Section II. We get following result. We skip the detail proof.

Theorem 14: Let G be a graph of order n, and let a, b, n', and m be non-negative integers such that such that a and b are two even integers with $2 \leq a \leq b$. If $\mathrm{bind}(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bn'-2m+3}$ and $n \geq \frac{(a+b)(a+b-3)}{b} + \frac{bn'+2m}{b-1}$, then G is a fractional (a, b, n', m)-critical deleted graph.

Theorem 15: Let a, b, n', n and m are non-negative integers such that a and b are two even integers with $2 \leq a \leq b$. Let G be a graph with order n such that $n \geq \frac{(a+b-1)(a+b-2)+bn'+2m-2}{b}$. If

$$|N_G(X)| > \frac{(a-1)n + |X| + bn' + 2m - 2}{a+b-1}$$

holds for each non-empty independent subset $X \subseteq V(G)$, and

$$\delta(G) > \frac{(a-1)n + a + b + bn' + 2m - 3}{a+b-1},$$

then G is a fractional (a, b, n', m)-critical deleted graph.

Theorem 16: Let G be a graph of order n, and let a, b, n', and m be non-negative integers such that such that a and b are two even integers with $2 \leq a \leq b$, $n \geq \frac{(a+b-1)(a+b-2)-2}{b} + \frac{bn'+2m}{b-1}$. If G satisfies

$$\mathrm{bind}(G) \geq \frac{(a+b-1)(n-1)}{b(n-1) - bn' - 2m + 1}$$

and

$$\delta(G) \neq \lfloor \frac{(a-1)n + a + b + bn' + 2m - 3}{a+b-1} \rfloor,$$

then G is a fractional (a, b, n', m)-critical deleted graph.

Again, similar as what we discussed before, Theorem 14 and Theorem 15 are sharp in some sense.

IV. CONCLUSION

In this paper, we present some results on the fractional (g, f, n', m)-critical graphs by bounding the isolated toughness and binding number. The new bound contributes to the state of art and the results achieved in our paper illustrates the promising application prospects for information transmission in networks when some channels and sites are destroyed.

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