

# A sampling method based on curvature analysis of cubic Bézier curve

Zhiyi Zhang<sup>\*a</sup>, Xian Zhang<sup>b</sup>, Huayang Zhang<sup>a</sup>

<sup>a</sup> College of Information Engineering, Northwest A&F University,  
No.22, XiNong Road, YangLing ShannXi 712100, China  
Email: 815802490@qq.com

<sup>b</sup> Technology of thermal power, Yunnan Power Technologies Inc,  
Kunming Yunnan 650200, China  
Email: liurenzu@sohu.com

**Abstract**—In order to eliminate the weakness of large volumes data in 3D modeling process, a fast and efficient sampling method based on curvature analysis of cubic Bézier curve is presented. With the method, the characteristic points and curvature maximum value of cubic Bézier curve are used as the reference standard of the sampling interval and density. For cubic B-spline curves, the sampling data can be obtained by dividing them into some cubic Bézier curves. Some experimental results show that this method is feasible and effective for the sampling of cubic free-form curves.

**Index Terms**—Cubic Bézier curve, Geometric Construction, Curvature Analysis, Cubic B-spline curve, Sampling

## I. INTRODUCTION

IN the medical image processing, geographic information system and reverse engineering applications, 3D shape modeling reconstruction from contours has been, and continues to be an important topic of research. Usually, the contours are expressed as a set of B-spline curves. As a prerequisite for 3D geometry reconstruction, obtaining the discrete sampling data points is necessary from initial B-spline curves. Sampling of data points determines the merits of the modeling reconstruction. However, it's difficult to express the sufficient conditions of sampling standard into formula or theorem. The inappropriate sampling data points may cause the following two problems. First, the geometric model cannot be effectively reconstructed or reconstruction distort when the sampling data points are too sparse. Second, when the sampling data points are too dense, the reconstruction process will increase the computational overhead, make computing time longer and take more memory space. For the above problems, Nina Amenta etc [1-2]. and M.Gopi etc [3]. proposed some sampling standards based on constrained condition of sampling density. In these papers, curvature analysis was the basis for sampling density, but the specific algorithm for the sampling standards was

proposed. Therefore, in that case, a concrete and effective sampling method is extremely important.

In this paper, a sampling method based on curvature analysis of cubic Bézier curve was first proposed. The rest of this paper is organized as follows: Section II described in details about how to obtain the characteristic points and curvature maximum value of cubic Bézier curve, how to determine the sampling standard and design algorithm for it. Section III explained how to design the sampling algorithm of cubic B-spline curve. Section IV listed some typical experimental results and analyzed the performance of the proposed algorithm. The discussion and conclusion was given in Section V.

## II. SAMPLING METHOD OF CUBIC BÉZIER CURVE AND B-SPLINE CURVE

In general, when the original data are scanned points, they can be fit into a set of cubic B-spline curves using the algorithm proposed by LesA. Piegl and Wayne Tiller [4-5]. However, owing to the multiple effects from degree, control polygon and knot vector, it's difficult to analyze the shape characteristics of B-spline curve. Fortunately, the general non-rational B-spline curve can be divided in to non-rational Bézier curve [6-7].

### A. The characteristic points of cubic Bézier curve

In the paper, inflection points are called shape characteristic points which contain the non-singular inflection points and singular points [6-7]. There are some different definitions in different fields about inflection points. In the Computer Aided Geometric Design field, inflection points are defined by the following **Theorem 1** and **Theorem 2**.

**Theorem 1:** A point on a curve where any of the neighborhoods around the point are at both sides of its tangent line. Its curvature, if exists, is zero [8].

**Theorem 2:** For a point  $\mathbf{P}$ , if the curvature radius of points  $\mathbf{P}_+$  and  $\mathbf{P}_-$  located on both sides of the curve, the point  $\mathbf{P}$  is a inflection point [9].

For curves on the  $\mathbf{XY}$  plane, curvature can be expressed by equation (1):

$$\kappa(t) = \frac{|\mathbf{C}'(t) \times \mathbf{C}''(t)|}{(|\mathbf{C}'(t)|)^3} \quad (1)$$

This work was supported by the National High Technology Research and Development Program of China (863 Program: No.2013AA10230402), Chinese Universities Scientific Fund (No.QN2013054), Shaanxi province science and technology plan project of China (No.2012NKC01-20).

According to **Theorem 1** and **Theorem 2**, the curvature of non-singular inflection points is zero. Considering the expression of curvature, only when the numerator of expression is zero, curvature is zero. When the denominator is zero, curvature does not exist and this point is called singular point. If the curve is cubic Bézier curve, it can be expressed as equation (2):

$$C(t) = \sum_{i=0}^3 B_i^3(t)P_i \quad t \in [0, 1] \quad (2)$$

Here,  $P_i$  are the control points, and they constitute control polygon of cubic B-spline curve.  $B_i^3(t)$  are cubic Bernstein basis function shown in the following equation (3):

$$B_i^3(t) = \frac{3!}{i!(3-i)!} t^i (1-t)^{3-i} \quad i = (0, \dots, 3) \quad (3)$$

Then, the first derivative and second derivative of cubic Bézier curve can be expressed by equation (4) and (5) respectively:

$$C'(t) = 3 \sum_{i=0}^2 B_i^2(t)(P_{i+1} - P_i) = 3(A_0 + 2tD_0 + t^2E_0) \quad (4)$$

$$C''(t) = 6((1-t)D_0 + tD_1) = 6(D_0 + tE_0) \quad (5)$$

In the equation (4) and (5):

$$\begin{aligned} A_i &= P_{i+1} - P_i & (i = 0, 1, 2) \\ D_j &= A_{j+1} - A_j & (j = 0, 1) \\ E_k &= D_{k+1} - D_k & (k = 0) \end{aligned} \quad (6)$$

Here,  $A_i$ ,  $D_j$  and  $E_k$  is first-order, second-order and third-order control vector of cubic Bézier curve respectively.

According to the theory of CAGD, Point  $P$  with parameter  $t$  on curve  $C(t)$  may be inflection points of curve when  $C'(t) \times C''(t) = 0$ . For vectors  $C'(t)$  and  $C''(t)$ , only two possibilities can result their cross product is 0:

- When  $C'(t) \times C''(t) = 0$  and  $C'(t) \neq 0$ : then  $\kappa(t) = 0$ .  $C''(t_-)$  and  $C''(t_+)$  should have the opposite direction for vector  $C'(t)$ . So  $P$  is called non-singular points. For a straight line  $C(t)$ , since  $C''(t) = 0$ , the curvature at each point of it is zero. In this paper, straight line is not to be considered in the scope.
- When  $C'(t) \times C''(t) = 0$  and  $C'(t) = 0$ : then  $\kappa(t) = \frac{0}{0}$ , the curvature of the point can not be calculated. For the continuity of  $C'(t)$ ,  $C''(t_-)$  and  $C''(t_+)$  there should be the opposite direction for vector  $C'(t)$ . In general, point  $P$  is called the curve of singular point if  $C'(t) = 0$ . So singular point is also special inflection point.

Use  $A_{0x}$ ,  $A_{0y}$ ,  $D_{0x}$ ,  $D_{0y}$ ,  $E_{0x}$ ,  $E_{0y}$  represents respectively the  $x$  and  $y$  components of vectors  $A_0$ ,  $D_0$  and  $E_0$ , if:

$$\begin{aligned} o &= A_{0x}D_{0y} - A_{0y}D_{0x} \\ p &= D_{0x}E_{0y} - D_{0y}E_{0x} \\ q &= A_{0x}E_{0y} - A_{0y}E_{0x} \end{aligned} \quad (7)$$

Combine equation (4), (5) and (7), necessary and sufficient conditions for inflection points existence can be obtained by equation (8):

$$pt^2 + qt + o = 0 \quad (t \in [0, 1]) \quad (8)$$

The parameter  $t \in [0, 1]$  can be computed by equation(9):

$$t = \frac{-q \pm \sqrt{q^2 - 4po}}{2p} \quad (t \in [0, 1]) \quad (9)$$

Especially, When  $C'(t) = 0$ , necessary and sufficient conditions for singular points it can be obtained by equation (10):

$$\begin{aligned} t &= \frac{-D_{0x} \pm \sqrt{D_{0x}^2 - A_{0x}E_{0x}}}{E_{0x}} \\ &= \frac{-D_{0y} \pm \sqrt{D_{0y}^2 - A_{0y}E_{0y}}}{E_{0y}} \end{aligned} \quad (10)$$

After the parameter  $t$  is obtained, use equation (2) to calculate the coordinate value of characteristic points of curve. According to the relative position relationship of the four control points for cubic Bézier curve, the control polygon will be one of the following three forms [6]:

- 1)  $P_i (i=0, \dots, 3)$  lying on the same line.
- 2)  $P_0, P_3$  lying on both sides of  $A_1$ .
- 3)  $P_0, P_3$  lying on the same side of  $A_1$ .

In order to find the shape characteristic points, the possible combination of curvature expression and control polygon shape will be discussed in detail. For the above three forms, Form 1 is often not taken into account. There must be at least one inflection point in Form 2. Singular points and non-singular inflection points might exist in Form 3. Then, a rapid extraction algorithm of shape characteristic points can be obtained as follows:

**Input:**

o: the value of cross product of vector  $A_0$  and  $D_0$ .

p: the value of cross product of vector  $D_0$  and  $E_0$ .

$C'(t)$ : the first derivative of cubic Bézier curve.

**Output:**

$T = (T_0, T_1)$ : the parameter of characteristic points.

**Algorithm:**

$A_{0x}$ : the X component of  $A_0$ ;  $A_{0y}$ : the Y component of  $A_0$ ;

$D_{0x}$ : the X component of  $D_0$ ;  $D_{0y}$ : the Y component of  $D_0$ ;

$E_{0x}$ : the X component of  $E_0$ ;  $E_{0y}$ : the Y component of  $E_0$ ;

$$t_0 = -q + \sqrt{q^2 - 4 * p * o} / 2 * p;$$

$$t_1 = -q - \sqrt{q^2 - 4 * p * o} / 2 * p;$$

$$T_0 = -D_{0x} + \sqrt{D_{0x} * D_{0x} - A_{0x} * E_{0x}} / E_{0x};$$

$$T_1 = -D_{0x} - \sqrt{D_{0x} * D_{0x} - A_{0x} * E_{0x}} / E_{0x};$$

$$\text{if}((t_0, t_1) \in [0, 1])$$

$$\text{if}(C'(t_0) == 0 \text{ or } C'(t_1) == 0 \text{ or } (C'(t_0) == 0$$

```

and C'(t1)=0)
    T0; T1;
else T0 = t0; T1 = t1;
else if((t0) ∈ [0, 1])
    if(C'(t0)=0)
        T0; T1;
    else T0 = t0;
else if((t1) ∈ [0, 1])
    if(C'(t1)=0)
        T0; T1;
    else T0 = t1;

```

**End**

**B. Sampling algorithm of cubic Bézier curve**

Any cubic Bézier curve with Form 2 can be divided at the characteristic points into two segments of cubic Bézier curves with Form 3 by de Casteljau [10] algorithm. The edges of control polygon of each cubic Bézier curve segment must not cross. Any cubic Bézier curve with Form 3 can be divided at the characteristic points into at least two segments of cubic Bézier curves with Form 3. Similarly, all control polygons edges of cubic Bézier curve segments must not cross. Each segment of cubic Bézier curves with Form 3 and their control polygon do not cross, the curvature change has only four situations shown in Figure 1.

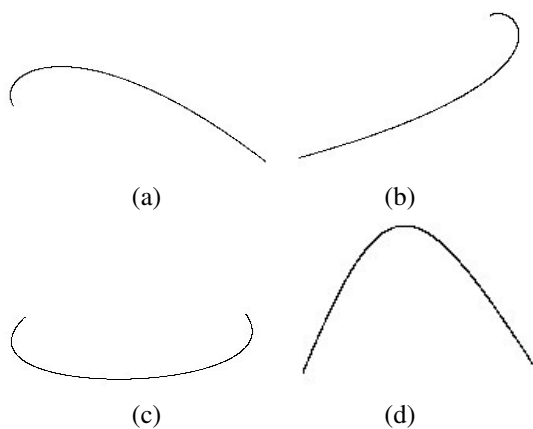


Figure 1. The situations of curvature change of cubic Bézier curve

In Figure 1, suppose the left is the start point of curves and the right is the end point of curves. Figure 1(a) shows that curvature changes from large to small, so  $(k(0) > k(1) \&\& k'(0) * k'(1) > 0)$  or  $(k(0) > k(1) \&\& k'(0) * k'(1) = 0)$ . Figure 1(b) shows the case that curvature changes from small to large, where  $(k(0) < k(1) \&\& k'(0) * k'(1) > 0)$  or  $(k(0) < k(1) \&\& k'(0) * k'(1) = 0)$ . Figure 1(c) shows that curvature changes from large to small then increasing, or from small to large, then becomes smaller, then bigger, if the change of curvature is the first case, the determine condition is  $(k(0) > k(\frac{1}{2}) \&\& k(1) > k(\frac{1}{2}))$ , if the change of curvature is the second case, it needs divide the curve until the change of curvature fits one of the other three cases. Figure 1(d) shows that curvature changes from

small to large then reduce, so  $(k(0) < k(\frac{1}{2}) \&\& k(1) < k(\frac{1}{2}))$ . That means when the change of curvature is the first or second case, the maximum curvature of curve appears the two endpoints of curve. When the change of curvature is the third or fourth case, the maximum of curvature can be approximated by dichotomy algorithm. When the curves are more than one, suppose the sampling density is  $d$ , the number of sampling data is  $N$ , the length of control polygon is  $L$ , the curvature radius of this curve is  $R$ , the interval between contours is  $l$ . So  $d = \min(R, l)$ ,  $N = \frac{L}{d}$ , and the change rate of parameter  $t$  is  $\frac{1}{N}$ . Sampling algorithm can be described as follows:

**Input:**

- sum: the number of contour of scanned object;
- r: the minimum curvature radius of cubic Bézier curve;
- interval: the interval between of contours;
- L: the length of a cubic Bézier curve;

**Output:**

- d: the sampling density;
- N: the number of sampling data;
- k: the change rata of parameter;

**Algorithm:**

```

if(sum>1)
    if (r>1e6) d= interval/4;
    else
        if (interval>r) d=r;
        else d=interval;
    N=L/d; k=d/L;
else
    a=sqrt(L); b=L/R;
    if(a>b) d=a;
    else d=b;
    N=L/d; k=d/L;

```

**End**

With the proposed sampling algorithm of curve, the least sampling data can be obtained while maintaining the curve shape.

**III. SAMPLING ALGORITHM OF CUBIC B-SPLINE CURVE**

When sampling B-spline curve, an effective method is: First, convert cubic B-spline curve into Bézier curve. Then use the algorithm mentioned in the previous section. A rapid and robust algorithm that can divide cubic B-spline curve into some segments of cubic Bézier curve is shown as follows:

**Input:**

- $P_i, i = (0, \dots, 3)$ : the control points of cubic B-spline curve;
- $U = u_0, \dots, u_{2n+k+1}$ : the knot vector of cubic B-spline curve;
- K: degree;
- t: the definite parameter,  $t \in [u_m, u_{m+1}), m \geq 3$ ;

**Output:**

$Q_i, i = (0, \dots, 3)$ :the control points of cubic Bézier curve;

$\mu$ : the parameter of new cubic Bézier curve;

**Algorithm:**

$\alpha = x_{m+1} - t; \beta = t - x_m;$   
 $\gamma = x_{m+2} - t; \delta = t - x_{m-1};$   
 $\epsilon = x_{m+3} - t; \eta = t - x_{m-2};$   
 $\lambda = x_{m+1} - x_m; \xi = x_{m+1} - x_{m-1};$   
 $\sigma = x_{m+1} - x_{m-2}; \varphi = x_{m+2} - x_m;$   
 $\psi = x_{m+2} - x_{m-1}; \omega = x_{m+3} - x_m;$   
 $\theta = x_{m+3} - x_{m+1}; \vartheta = x_{m+2} - x_{m+1};$   
 $\nu = x_m - x_{m-1}; \varpi = x_m - x_{m-2};$

$$Q_0 = \frac{\lambda^2}{\sigma\xi}P_{m-3} + \left(\frac{\nu\varphi}{\psi\xi} + \frac{\varpi\lambda}{\sigma\xi}\right)P_{m-2} + \frac{\nu^2}{\psi\xi}P_{m-1}$$

$$Q_1 = \left(\frac{\varphi}{3\psi} + \frac{2\lambda}{3\xi} + \frac{2\vartheta\nu}{3\xi\psi}\right)P_{m-2} + \frac{\nu}{\psi}P_{m-1}$$

$$Q_2 = \left(\frac{\xi}{3\psi} + \frac{2\lambda}{3\varphi} + \frac{2\vartheta\nu}{3\varphi\psi}\right)P_{m-1} + \frac{\vartheta}{\psi}P_{m-2}$$

$$Q_3 = \frac{\vartheta^2}{\varphi\psi}P_{m-2} + \left(\frac{\xi\vartheta}{\varphi\psi} + \frac{\lambda\theta}{\varphi\omega}\right)P_{m-1} + \frac{\lambda^2}{\varphi\omega}P_m$$

$$\mu = \frac{\beta}{\lambda}$$

**End**

The difference between B-spline and Bézier is that the points corresponding to multiplied knot vectors are considered as shape characteristic points of curve.

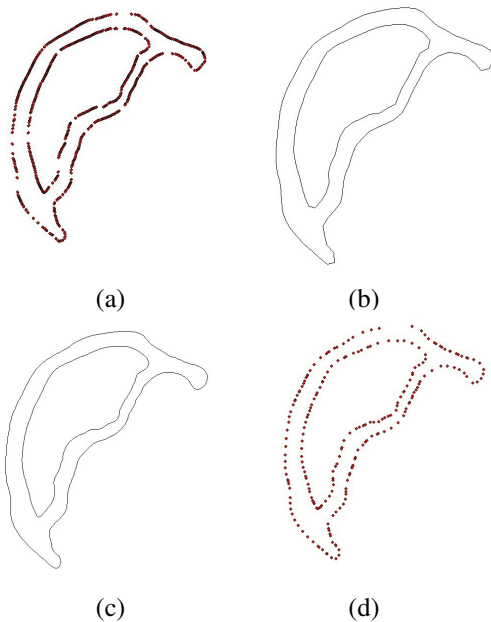


Figure 2. A mountain contours data: (a) The scanned data, (b) Cubic B-spline curve, (c) Cubic Bézier curve, (d) The sampling result.

**IV. EXPERIMENTAL RESULT**

Experimental result contains non-uniform cubic periodic B-spline curve and non-periodic cubic B-spline curve.

Figure 2 and Figure 3 show the sampling result of periodic cubic B-spline curve for non-uniform knot vectors.

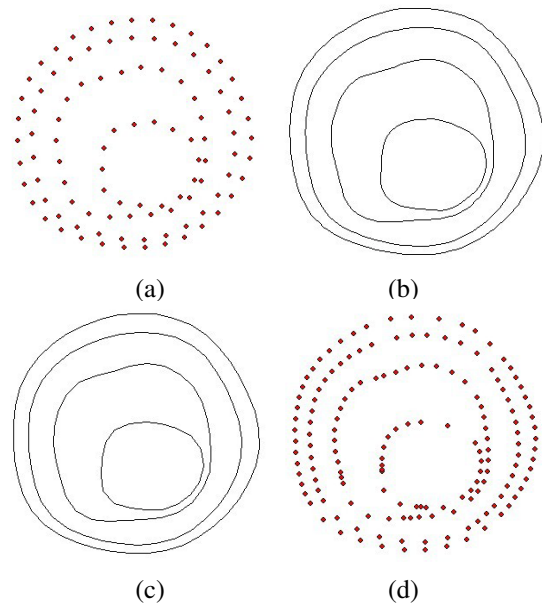


Figure 3. A pumpkin contours data: (a) The scanned data, (b) Cubic B-spline curve, (c) Cubic Bézier curve, (d) The sampling result.

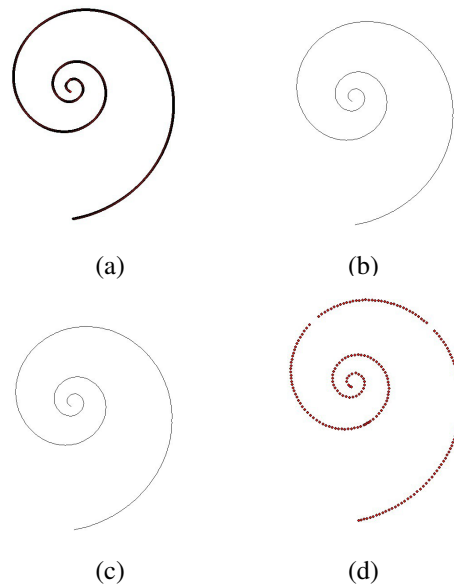


Figure 4. A logarithmic spiral: (a) The source data, (b) Cubic B-spline curve, (c) Cubic Bézier curve, (d) The sampling result.

Figure 4 shows the sampling result of non-periodic cubic B-spline for non-uniform knot vector.

Figure 5 shows an application of curvature analysis in offset curve [11], and the interval between cubic Bézier curve and offset curve depends on the radius of curvature, the details were described: the original interval is  $d$ , the radius of curvature is  $r$ , and the last interval is  $dd$ . If  $r > d$  or  $r = d$ ,  $dd = d$ , else  $dd = r * r/d$ . The red curve showed the cubic Bézier curve, the green curve showed the gradient offset curve of the cubic Bézier curve, and the original interval is 25 in Figure 5. It can be found that the offset curve is gradually changing with the change of curvature, and rapidly changing in the place of the

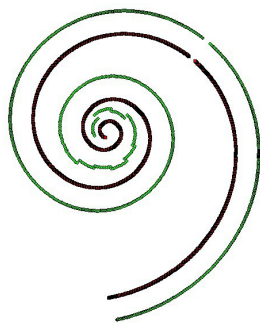


Figure 5. The application of curvature analysis in offset curve (the original interval is 25).

TABLE I.  
THE COMPARISON OF THE NUMBER OF SCANNED DATA AND SAMPLING DATA

Basis information	scanned points	sampling points
Fig2	(638),(468)	(155),(107)
Fig3	(14),(23),(29),(35)	(25),(43),(46),(51)
Fig6	(1460)	(233)

maximum curvature.

Figure 6 shows an example of extracting characteristic points and the curvature extremum points. It is clear that the shape of the word "Yong" can be described completely by these points from Figure 6, the green solid round expresses the points that have maximum value of curvature, and red solid rectangle expresses the characteristic points.

Table I shows the comparison of the number of scanned data and sampling data from Figure 2, Figure 3 and Figure 6. In Table I, the amount of scanned data is less than that scanning sampling data in Figure 3, because the scan instrument is not accurate which results in the reduction in the number of scanned data, the fitting curve and sampling result are accurately described in the shape of pumpkin.

V. DISCUSSION AND CONCLUSION

A sampling method based on curvature analysis was proposed on the basis of analysis of cubic Bézier curve. Considering the characteristic points of cubic Bézier curve, a sampling standard for 3D geometry reconstruction was offered. The sampling standard can not only fulfill little calculation and small storage requirements, also can keep the shape of original curves invariable. Experimental results show the algorithm is feasible and effective and can be used for the data sampling of 3D geometry reconstruction. If the proposed algorithm was introduced into the application of [12-14], the effect will be better.

ACKNOWLEDGMENT

The authors are grateful to the anonymous referees for their valuable comments and suggestions to improve the presentation of this paper.

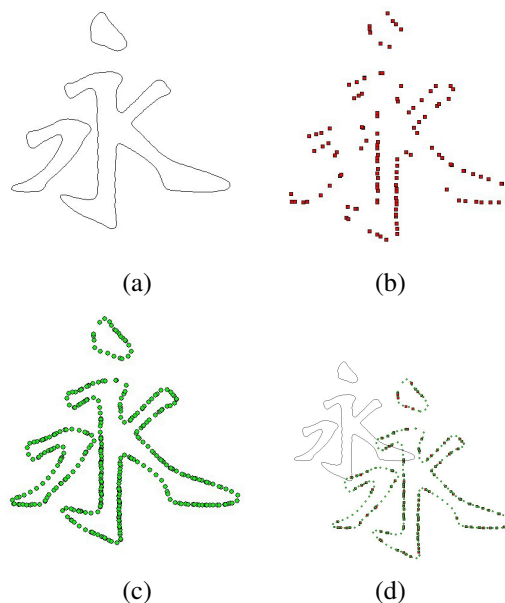


Figure 6. The contours of Chinese characters "Yong": (a) The cubic Bézier curve, (b) The characteristic points, (c) The curvature extremum points, (d) the comparison of three results.

REFERENCES

- [1] Nina Amenta, Marsahll Bern, Surface Reconstruction by Voronoi filtering. In ACM Symposium on Computational Geometry, (1998) 39-48.
- [2] Nina Amenta, Marsahll Bern and Manolis Kamvysselis, A New Voronoi-Based Surface Reconstruction Algorithm. In Proceedings of ACM Siggraph, (1998) 415-421.
- [3] M Gopi, S. Krishnan, C.T. Silva, Surface Reconstruction based on Lower Dimensional Localized Delaunay Triangulation. EUROGRAPHICS, 19(3) (2000) 467-478.
- [4] Piegl L, Tiller W., The NURBS book. NewYork: Springer, (1997).
- [5] Piegl L, Tiller W., Surface approximation to scanned data. The Visual Computer, 16(7) (2000) 386-395.
- [6] Zhiyi Zhang, Min Chen, Xian Zhang, Zepeng Wang, Analysis of Inflection Points for Planar Cubic Bezier Curve. International Conference on Computational Intelligence and Software Engineering, (2009).
- [7] Zhiyi Zhang, Xian Zhang, Sulei Tian, Min Chen, Extract Shape Characteristic Points From Cubic B-spline Curve By Segmented Cubic Bezier Curve. The International Conference on Multimedia Technology, 2(2010) 768-773.
- [8] Liu Chaoyang, Theory and Application of Convex Curves and Surfaces in CAGD, Faculty of Mathematical Science University of Twente, (2001).
- [9] Orzech G, Orzech M, Plane Algebraic Curves. Marcel Dekker, Inc, New York, (1981).
- [10] Gerald Farin, Curves And Surfaces For Computer Aided Geometric Design-A Practical Guide. Academic Press, Inc. (1997).
- [11] Gershon Elber, In-Kwon Lee, Myung-Soo Kim, Comparing Offset Curve Approximation Methods. Computer Graphics and Applications, 17(3) (1997) 62-71.
- [12] Jincal Chang, Qianli Yang, Long Zhao, Comparison of B-spline Method and Finite Difference Method to Solve BVP of Linear ODEs. JOURNAL OF COMPUTERS, 6(10) (2011) 2149-2155.
- [13] Shang Gao, Zaiyue Zhang, Cungen Cao, Differentiation and Numerical Integral of the Cubic Spline Interpolation. JOURNAL OF COMPUTERS, 6(10) (2011) 2037-2044.

- [14] Tunhua Wu, Qinqin Shen, Changle Zhou, Ping Wang, Design and Implementation of a General Purpose 2D CAD System. JOURNAL OF COMPUTERS, 7(3) (2012) 666-671.

**Zhiyi Zhang** received his B.S. degree ShanXi University, China in June 1998. His M.S. degree and PhD. in Electronic and Information Engineering from Graduate School of Engineering, Iwate University, Japan in 2004 and 2007, respectively. He is a associate professor of College of Information Engineering, Northwest A&F University, China. His research interests include Computer Aided Design, Computer Aided Geometric Design and Computer Graphics.

**Xian Zhang** received her B.S. and M.S. degree in Information Engineering from College of Information Engineering, Northwest A&F University, China in 2009 and 2012, respectively. She is a software development engineer working in Technology of thermal power of Yunnan Power Technologies Inc. Her research interests include Computer Aided Design, Computer Aided Geometric Design and Computer Graphics.

**Huayang Zhang** received his B.S. degree in College of Computer Science and technology, Henan Polytechnic University, China in 2011. He is Master student of College of Information Engineering, Northwest A&F University and His research interests include Computer Aided Design and Computer Aided Geometric Design.