Bargmann and Neumann System of the Second-Order Matrix Eigenvalue Problem

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***Abstract*—This paper discusses the second-order matrix eigenvalue problem by means of the nonlinearization of the Lax pairs, then the author gives the Bargmann and Neumann constraints of this problem. The relation between the potentials and the eigenfunction is set up based on these constraints. By means of the nonlinearization of the Lax pairs, the author found these systems of the eigenvalue problem can be equal to the Hamilton canonical system in real symplectic space. In the end, the infinite-dimensions Dynamical systems can be transformed into the finite-dimensions Hamilton canonical systems in the symplectic space. As well, this paper obtains the representations of the solutions for the evolution equations.**

***Index Terms*—eigenvalue problem, integrable system, evolution equation, lax representation**

I. INTRODUCTION

The understanding of completely integrable Hamilton system experienced a tortuous process. To plot out the kinematic equations is the highest goal of the early classical mechanics. With the rise of the solitary, a lot of nonlinear evolution equations have been proven restricted submanifold in completely integrable system. Thus the investigation for them is one of the most highlighted subjects in mathematical physics. The integrable system is widely used in fluid mechanics, nonlinear optics and a series of nonlinear science. To find as much as possible of the integrable system and study its algebra, geometry and other properties can judge whether a system is completely integrable or not, and it can be applied to nonlinear evolution equation. As everyone knows, the famous Liouville theorem has laid a finite dimensional Hamilton completely integrable system beautiful geometric theory. According to Liouville theorem, if a 2n dimensional Hamilton system has n of independent and dual involution on the conservation integral [1, 2], then for the infinite dimension the situation is dimensional completely integrable has an infinite number of independent and dual involution on the conservation integral. In the past, systematic methods were developed to study soliton equations such as the inverse scattering, Darboux transformation, etc. Finitedimensional completely integrable systems are closely connected with infinite dimensional completely integrable systems described by soliton equations. For instance, when studying of pole solutions of soliton equations, one found that the equations of motion of poles are finite-dimensional integrable systems. More examples are that the stationary equations of soliton equations are finite-dimensional integrable systems. Initial dimensional integrable systems can generate from finite dimensional integrable systems. The resulting infinite dimensional integrable systems are mainly divided into two kinds: one is the Bargmann type in the freedom, and the other is the Neumann type with an additional constraint. The former has been investigated by many authors in a series of papers. By the contrast, the later is more difficult to be treated owning to an additional constraint, and few examples were discussed such as the cKDV. To our opinion, the Dirac bracket and the Moser's constraint are effective in exploring the Neumann type finite dimensional integrable systems, so-called Lax pairs or nonlinearization of eigenvalue problems.

Therefore, to seek a new completely integrable system which is associated with the development of non-linear equations is an interesting issue in the international mathematical physics union. This paper has obtained a new finite-dimensional completely integrable system with the nonlinear of the eigenvalue problem.

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II. LAX REPRESENTATION AND THE EVOLUTION EQUATION HIERARCHY RELATED TO EIGENVALUE PROBLEM

TABLE I. Note symbols

NO	NOTE SYMBOLS	
	symbols	note
1	$\partial = \frac{\partial}{\partial x}$	Partial derivative of x
2	Λ	Diagonal line of matrix is $(\lambda_1, \lambda_2 \cdots \lambda_n)$
3	φ^T	Vector transformation of φ
4	$\langle \varphi, \Psi \rangle = \sum_{k=1}^{n} \varphi_k \Psi_k$	N dimensional vector inner-product of φ and Ψ
5	$[W, L] = WL - LW$	Differential operator of the lie algebra of transposition operation
6	$\nabla \lambda$	Functional gradient of λ

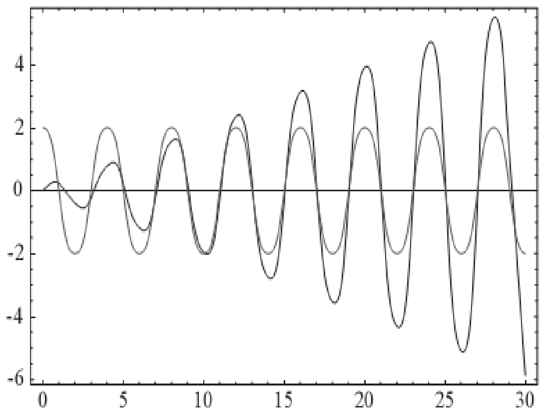

Figure1. The Jacobi fields

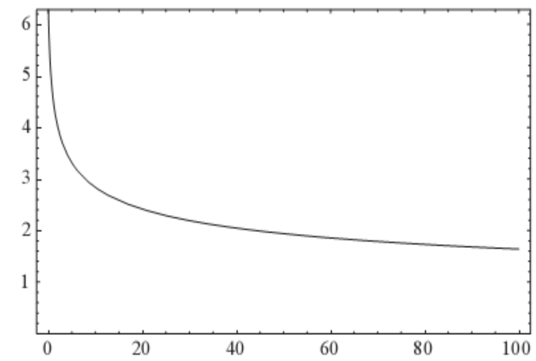

Figure2. The period of unit mass in the potentials plotted versus energy

This paper considers the second order matrix eigenvalue problem:

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}_x = \begin{pmatrix} \lambda - q & \lambda \\ \lambda + r & q - \lambda \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \qquad (1)$$

$$= M\varphi,$$

$u = \begin{pmatrix} r \\ q \end{pmatrix}$,Let $W = \sum_{j=0}^{\infty} \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \lambda^{-j}$,

from $W_x = [M, W]$,

$$w_{11x} - \lambda w_{21} + \lambda w_{12} + r w_{12} = 0$$

$$w_{12x} - 2\lambda w_{12} + 2q w_{12} - \lambda w_{22} + \lambda w_{11} = 0$$

$$w_{21x} + 2\lambda w_{21} - 2q w_{21} + \lambda w_{22} - \lambda w_{11} - r w_{11} + r w_{22} = 0$$

$$w_{22x} + \lambda w_{21} - r w_{12} - \lambda w_{12} = 0$$

It can get the following result:

$$w_{11} = -q a_{j-1} - \frac{1}{2} a_{j-1x} + \lambda a_{j-1},$$

$$w_{12} = \lambda a_{j-1},$$

$$w_{21} = -q a_{j-1} - \frac{1}{2} b_{j-1x} + \lambda a_{j-1}$$

$$w_{22} = q a_{j-1} + \frac{1}{2} a_{j-1x} - \lambda a_{j-1},$$

Definition:

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

$$K_{11} = \partial r + r\partial, \; K_{12} = q\partial - \frac{1}{2}\partial^2$$

$$K_{21} = \partial q + \frac{1}{2}\partial^2, \; K_{22} = 0$$

$$J = \begin{pmatrix} -2\partial & \partial \\ \partial & \frac{1}{2}\partial \end{pmatrix}, \quad K G_{j-1} = J G_j, \; j = 0, 1, 2, 3, \ldots \qquad (2)$$

$$G_j = \begin{pmatrix} a_j \\ b_j \end{pmatrix} \quad j = -1, 0, 1, 2, \cdots \qquad (3)$$

Note: Operators K, J are double *Hamilton* operator [3] it is that K, J have the Properties of antisymmetry, bilinear, non-degeneracy, and satisfy the Jacobi equation.

The definition of second-order matrix as follows

$$W_m = \sum_{j=0}^{m} \begin{pmatrix} -q a_{j-1} - \frac{1}{2} a_{j-1x} + \lambda a_{j-1} & \lambda a_{j-1} \\ -q a_{j-1} - \frac{1}{2} b_{j-1x} + \lambda a_{j-1} & q a_{j-1} + \frac{1}{2} a_{j-1x} - \lambda a_{j-1} \end{pmatrix} \lambda^{m-j}$$

This paper has the following proposition:

Proposition 2.1: The Evolution Equation [4] Hierarchy related to eigenvalue problem is ($u = (r, q)^T$):

$$u_{tm} = \begin{pmatrix} r_{tm} \\ q_{tm} \end{pmatrix} = J G_m, \; m = 0, 1, 2, \cdots \ldots \qquad (4)$$

$$M_{tm} = (w_m)_x + w_m M - M w_m \qquad (5)$$

In other words, (4) is consistency condition of spectrum-preserving for the following two linear equations:

$$(\varphi_{x t_m} = \varphi_{t_m x}, \lambda_{t_m} = 0)$$

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}_x = M\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$
$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}_{tm} = W_m\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \quad (6)$$

Let

$$JG_{-1} = 0\,,$$

so $G_{-1} = \begin{pmatrix}4\\0\end{pmatrix}$, from $KG_{m-1} = JG_m$,

$$G_0 = \begin{pmatrix}2q - r\\ 4q + 2r\end{pmatrix} \quad (7)$$

$$G_1 = \begin{pmatrix}4rq_x + 4qr_x - 3rr_x + 4qq_x - r_{xx} - 2q_{xx}\\ 4qq_x - \frac{1}{2}r_{xx} + q_{xx} - (qr)_x\end{pmatrix}$$

Then this paper has the evolution equation:

$$u_{t0} = \begin{pmatrix}r_{t0}\\ q_{t0}\end{pmatrix} = \begin{pmatrix}4r_x\\ 4q_x\end{pmatrix} \quad (8)$$

$$u_{t1} = \begin{pmatrix}r_{t1}\\ q_{t1}\end{pmatrix}$$
$$= \begin{pmatrix}4qr_x + 4rq_x - 3rr_x - r_{xx} - 2q_{xx}\\ 4qq_x + (qr)_x + q_{xx} - \frac{1}{2}r_{xx}\end{pmatrix} \quad (9)$$

$$u_{t2} = \begin{pmatrix}r_{t2}\\ q_{t2}\end{pmatrix}$$

$$r_{t2} = 3qq_x r - 3r^2q_x - 6qrr_x + \frac{3}{2}rq_{xx} + \frac{15}{8}r^2r_x$$
$$+\frac{3}{2}q^2r_x + \frac{3}{4}r_x^2 + 6q^2q_x x - \frac{3}{2}qr_{xx}$$
$$-3qq_{xx} - 3q_x^2 + \frac{3}{4}rr_{xx} + \frac{1}{2}r_{xxx} - \frac{1}{2}q_{xxx}$$

$$q_{t2} = -3qq_x r + \frac{3}{2}q^2q_x - \frac{3}{2}q^2r_x +$$
$$\frac{3}{4}qrr_x + \frac{3}{2}q_x^2 + \frac{3}{8}r^2q_x - \frac{3}{4}q_{xx}r - \frac{3}{2}q_xr_x - \frac{3}{4}qr_{xx}$$
$$+\frac{1}{2}q_{xxx} + \frac{3}{8}r_x^{\,2} + \frac{3}{8}rr_{xx}$$

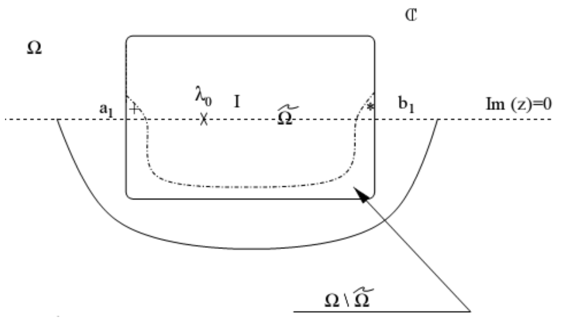

Figure3. Sample results

Proposition 2.2: Let $\varphi = (\varphi_1, \varphi_2)^T$ If the trace [5] of the second- order matrices M over the real is 0, so

$$\int_\Omega (\varphi_1,\varphi_2)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\dot{M}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}dx = 0$$

the Functional derivation is denoted by Symbol $\bullet$.

Let Ω be an open relatively compact subset. We assume that I and J are intervals. See Figure 3.

Proposition 2.3: If φ_1, φ_2 is the eigenfunction of the eigenvalue problem (1) relates to λ , so the functional gradient [6, 7] as follows:

$$grad\,\lambda = \begin{pmatrix}\frac{\delta\lambda}{\delta r}\\ \frac{\delta\lambda}{\delta q}\end{pmatrix} = \begin{pmatrix}\varphi_1^2\\ 2\varphi_1\varphi_2\end{pmatrix},$$

Proof: $\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}_x = \begin{pmatrix}\lambda - q & \lambda\\ \lambda + r & q - \lambda\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$

$$= M\varphi,$$

$$\int_\Omega (\varphi_1,\varphi_2)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\dot{M}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}dx = 0\,,$$

$$\int_\Omega (\varphi_1,\varphi_2)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\delta\lambda - \delta q & \delta\lambda\\ \delta\lambda + \delta r & -\delta\lambda + \delta q\end{pmatrix}\varphi dx = 0$$

$$\int_\Omega [(2\varphi_1\varphi_2 - \varphi_1^{\,2} + \varphi_2^{\,2})\delta\lambda - 2\varphi_1\varphi_2\delta q - \varphi_1^{\,2}\delta r]dx = 0$$

The eigenfunction is standard, $(\int_\Omega (2\varphi_1\varphi_2 - \varphi_1^{\,2} + \varphi_2^{\,2})dx)^{-1} = 1$, so conclusion correct.

Proposition 2.4: Functional gradient satisfying the following equations

$$K\nabla\lambda = \lambda J\nabla\lambda\,.$$

Proof: $K\nabla\lambda =$

$$\begin{pmatrix}\partial r + r\partial & q\partial - \frac{1}{2}\partial^2\\ \partial q + \frac{1}{2}\partial^2 & 0\end{pmatrix}\begin{pmatrix}\varphi_1^{\,2}\\ 2\varphi_1\varphi_2\end{pmatrix} =$$

$$\begin{pmatrix}2\lambda r\varphi_1^{\,2} + 4\lambda q\varphi_1^{\,2} - 4\lambda^2\varphi_1\varphi_2 - 2\lambda^2\varphi_1^{\,2} + 2\lambda^2\varphi_2^{\,2}\\ -2\lambda q\varphi_1^{\,2} + 2\lambda^2\varphi_1\varphi_2 + 3\lambda^2\varphi_2^{\,2} + 4\lambda^2\varphi_1^{\,2}\end{pmatrix}$$

$$\lambda J\nabla\lambda =$$

$$\begin{pmatrix}-2\partial & \partial\\ \partial & \frac{1}{2}\partial\end{pmatrix}\begin{pmatrix}\varphi_1^{\,2}\\ 2\varphi_1\varphi_2\end{pmatrix} =$$

$$\begin{pmatrix}2\lambda r\varphi_1^{\,2} + 4\lambda q\varphi_1^{\,2} - 4\lambda^2\varphi_1\varphi_2 - 2\lambda^2\varphi_1^{\,2} + 2\lambda^2\varphi_2^{\,2}\\ -2\lambda q\varphi_1^{\,2} + 2\lambda^2\varphi_1\varphi_2 + 3\lambda^2\varphi_2^{\,2} + 4\lambda^2\varphi_1^{\,2}\end{pmatrix}$$

so $K\nabla\lambda = \lambda J\nabla\lambda$

III. The Hamilton Equation and Its Complete Integrability Under the Bargmann Constraint

Suppose that $\lambda_1 < \lambda_2 < \cdots \lambda_N$ is N different eigenvalues of (5). $\varphi_{1j}, \varphi_{2j}$ are characteristic function of λ_j ($j = 1, 2, \cdots, N$), so

$$\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\end{pmatrix}_x = M(u, \lambda_j)\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\end{pmatrix}, \quad (10)$$

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}_{t_m} = w_m(u,\lambda_j)\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}[5], \quad (11)$$

Let $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_N)$,

$$\varphi_{1j} = (\varphi_{11}, ..., \varphi_{1N})^T,$$

$$\varphi_{2j} = (\varphi_{21}, ..., \varphi_{2N})^T,$$

From $Kgrad\lambda = \lambda Jgrad\lambda$, the result:

$$K\begin{pmatrix} \langle \Lambda^k \varphi_1, \varphi_1 \rangle \\ \langle \Lambda^k \varphi_1, \varphi_2 \rangle \end{pmatrix} = J\begin{pmatrix} \langle \Lambda^{k+1} \varphi_1, \varphi_1 \rangle \\ \langle \Lambda^{k+1} \varphi_1, \varphi_2 \rangle \end{pmatrix} \quad (12)$$

From (7) and (11),

$$G_j = \begin{pmatrix} \langle \Lambda^j \varphi_1, \varphi_1 \rangle \\ \langle \Lambda^j \varphi_1, \varphi_2 \rangle \end{pmatrix}, \ j = 0,1,2,\cdots$$

$$\begin{cases} a_{j-1} \\ b_{j-1} \end{cases} = \begin{pmatrix} \langle \Lambda^j \varphi_1, \varphi_1 \rangle \\ \langle \Lambda^j \varphi_1, \varphi_2 \rangle \end{pmatrix}, \ j = 1,2,\cdots \quad (13)$$

The constraint condition is Bargmann constraint condition:

$$G_0 = \begin{pmatrix} 2q - r \\ 4q + 2r \end{pmatrix} = \begin{pmatrix} \langle \varphi_1, \varphi_1 \rangle \\ \langle \varphi_1, \varphi_2 \rangle \end{pmatrix},$$

$$\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\langle \varphi_1, \varphi_2 \rangle + \frac{1}{4}\langle \varphi_1, \varphi_1 \rangle \\ \frac{1}{2}\langle \varphi_1, \varphi_2 \rangle - \frac{1}{2}\langle \varphi_1, \varphi_1 \rangle \end{pmatrix} \quad (14)$$

The Poisson bracket [8]of Smooth Function H and F in symplectic space $(R^{2N}, \sum_{k=1}^{N} d\varphi_{2k} \Lambda d\varphi_{1k})$ is defined as followed:

$$\{H, F\} = \sum_{k=1}^{N}\left(\frac{\partial H}{\partial \varphi_{2k}} \frac{\partial F}{\partial \varphi_{1k}} - \frac{\partial H}{\partial \varphi_{1k}} \frac{\partial F}{\partial \varphi_{2k}} \right)$$

Proposition 3.1: (10) and (11) can be written as a finite-dimensional system:

$$\begin{cases} \varphi_{1x} = \dfrac{\partial H}{\partial \varphi_2} & \varphi_{2x} = -\dfrac{\partial H}{\partial \varphi_1} \\ \varphi_{1t_m} = \dfrac{\partial H_m}{\partial \varphi_2} & \varphi_{2t_m} = -\dfrac{\partial H_m}{\partial \varphi_1} \end{cases}$$

$$m = 1,2,\cdots \quad (15)$$

H and H_m are *Hamilton* function here.

$$H = -\frac{1}{4}\langle \Lambda\varphi_1, \varphi_1 \rangle \langle \varphi_1, \varphi_2 \rangle - \frac{1}{8}\langle \Lambda\varphi_1, \varphi_2 \rangle \langle \varphi_1, \varphi_2 \rangle + \langle \Lambda\varphi_1, \varphi_2 \rangle + \frac{1}{2}\langle \Lambda\varphi_2, \varphi_2 \rangle + \frac{1}{8}\langle \Lambda\varphi_1, \varphi_1 \rangle \langle \varphi_1, \varphi_1 \rangle - \frac{1}{2}\langle \Lambda\varphi_1, \varphi_1 \rangle \quad (16)$$

$$H_m = -\frac{1}{2}\langle \Lambda^m \varphi_1, \varphi_1 \rangle \langle \varphi_2, \varphi_2 \rangle - 2\langle \Lambda^{m+1} \varphi_1, \varphi_1 \rangle + \frac{1}{2}\langle \Lambda^m \varphi_1, \varphi_1 \rangle \langle \varphi_1, \varphi_1 \rangle + 4\langle \Lambda^{m+1} \varphi_1, \varphi_2 \rangle + 2\langle \Lambda^{m+1} \varphi_2, \varphi_2 \rangle - \langle \Lambda^m \varphi_1, \varphi_1 \rangle \langle \varphi_1, \varphi_2 \rangle + \frac{1}{2}\sum_{j=0}^{m} \begin{vmatrix} \langle \Lambda^j \varphi_2, \varphi_2 \rangle & \langle \Lambda^j \varphi_2, \varphi_1 \rangle \\ \langle \Lambda^{m-j} \varphi_2, \varphi_1 \rangle & \langle \Lambda^{m-j} \varphi_1, \varphi_1 \rangle \end{vmatrix} \quad (17)$$

The generator E_k is as followed:

$$E_k = \frac{1}{2}\lambda_\kappa \Gamma_k - \varphi_{1k}^2 \langle \varphi_1, \varphi_2 \rangle + 4\lambda_\kappa \varphi_{1k}\varphi_{2k} + \frac{1}{2}\varphi_{1k}^2 \langle \varphi_1, \varphi_1 \rangle + 2\lambda_\kappa \varphi_{2k}^2 - \frac{1}{2}\varphi_{1k}^2 \langle \varphi_2, \varphi_2 \rangle$$

among them, $\Gamma_k = \sum_{\substack{j=1 \\ j \neq k}}^{N} \frac{(\varphi_{1k}\varphi_{2j} - \varphi_{2k}\varphi_{1j})^2}{\lambda_k - \lambda_j}$.

Proposition 3.2: (1) $\{E_k, k = 1,2,...N\}$ is involutive system: so that

$$\{E_k, E_j\} = 0, \forall k, j = 1,2,...N,$$

And $\{dE_k\}$ has nothing to do with the gradient;

(2) $H_\lambda = \sum_{k=1}^{N} \frac{1}{\lambda - \lambda_k} E_k = \sum_{m=0}^{\infty} \lambda^{-m-1} H_m$

$$H_m = \sum_{k=1}^{N} \lambda_\kappa{}^m E_k [9], m = 1,2,\cdots$$

Proof (1): From the define of Poisson bracket define,

$$\langle \varphi_{1k}, \varphi_{1l} \rangle = \langle \varphi_{2k}, \varphi_{2l} \rangle = 0$$

$$\langle \varphi_{1k}, \varphi_{2l} \rangle = \delta_{kl} = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}$$

$$(\langle \varphi_1, \varphi_1 \rangle, B_{kj}) = (\langle \varphi_1, \varphi_2 \rangle, B_{kj}) = (\langle \varphi_2, \varphi_2 \rangle, B_{kj}) = 0$$

$$(\Gamma_k, \Gamma_l) = (\sum_{\substack{j=1 \\ j \neq k}}^{N} \frac{(\varphi_{1k}\varphi_{2j} - \varphi_{2k}\varphi_{1j})^2}{\lambda_k - \lambda_j}, \sum_{\substack{j=1 \\ j \neq l}}^{N} \frac{(\varphi_{1l}\varphi_{2j} - \varphi_{2l}\varphi_{1j})^2}{\lambda_l - \lambda_j}) = 0,$$

$$(\varphi_{1k}^2, \Gamma_l) = (\varphi_{1k}^2, \sum_{\substack{j=1 \\ j \neq l}}^{N} \frac{(\varphi_{1l}\varphi_{2j} - \varphi_{2l}\varphi_{1j})^2}{\lambda_l - \lambda_j}) = 4(\lambda_k - \lambda_l)^{-1} \varphi_{1k}\varphi_{1l} B_{lk}$$

$$(\varphi_{2k}^2, \Gamma_l) = (\varphi_{2k}^2, \sum_{\substack{j=1 \\ j \neq l}}^{N} \frac{(\varphi_{1l}\varphi_{2j} - \varphi_{2l}\varphi_{1j})^2}{\lambda_l - \lambda_j}) = 4(\lambda_k - \lambda_l)^{-1} \varphi_{2k}\varphi_{2l} B_{kl}$$

$$(\varphi_{1k}\varphi_{2k}, \Gamma_l) = 2(\lambda_k - \lambda_l)^{-1}(\varphi_{2k}\varphi_{1l} + \varphi_{1k}\varphi_{2l}) B_{lk}$$

so $(E_k, E_l) = 0$

Proof (2): $\sum_{k=1}^{n} \frac{E_k}{\lambda - \lambda_k} = \sum_{k=1}^{n} \frac{1}{\lambda} \frac{E_k}{1 - \frac{\lambda_k}{\lambda}} E_k = \sum_{k=1}^{n} \frac{1}{\lambda} \sum_{m=0}^{\infty} \left(\frac{\lambda_k}{\lambda} \right)^m E_k$

$$= \sum_{m=0}^{\infty} \lambda^{-m-1} \sum_{k=1}^{n} (\lambda_k)^m E_k = \sum_{m=0}^{\infty} \lambda^{-m-1} H_m \ [10]$$

Proposition 3.3: (1) $\{H, E_k\} = 0, k = 1,2,...N.$

(2) $\{H, H_m\} = 0, m = 0,1,2,...$

(3) $\{H_n, H_m\} = 0, m, n = 0,1,2,...$

(4) $(R^{2n},\sum_{k=1}^{n} d\varphi_{1k}\Lambda d\varphi_{1k},H)$,

$(R^{2n},\sum_{k=1}^{n} d\varphi_{1k}\Lambda d\varphi_{1k},H)$

is the completely integrable system in the Liouville sense.

Proof: $\{H,E_k\}=0,k=1,2,...N$. can be proved by calculation. $\{H,H_m\}=0,m=0,1,2,...$; $\{H_n,H_m\}=0,m,n=0,1,2,...$ can be proved by the previous proposition. $(R^{2n},\sum_{k=1}^{n} d\varphi_{1k}\Lambda d\varphi_{1k},H)$, $(R^{2n},\sum_{k=1}^{n} d\varphi_{1k}\Lambda d\varphi_{1k},H)$ is proved by Liouville theorem. Hence the proposition is concluded.

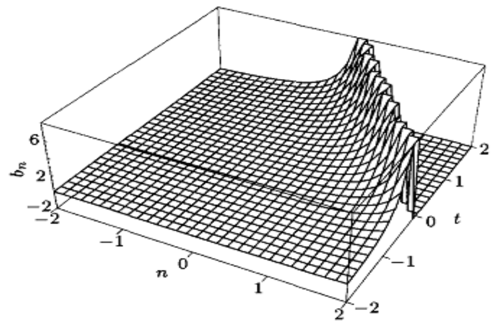

Figure4. Solition solution

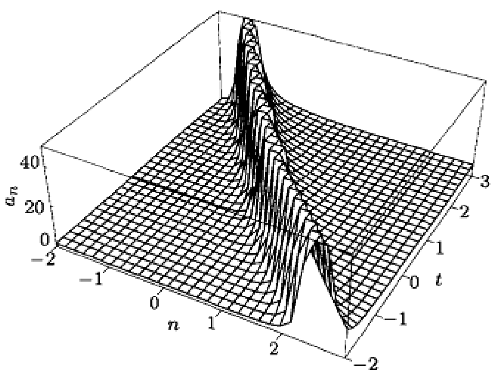

Figure5 Solition solution

Proposition 3.4: if φ_1,φ_2 are involutive solution of the Hamilton regular system [11], so

$$\begin{pmatrix} r \\ q \end{pmatrix}=\begin{pmatrix} \frac{1}{2}\langle\varphi_1,\varphi_2\rangle-\frac{1}{2}\langle\varphi_1,\varphi_1\rangle \\ \frac{1}{4}\langle\varphi_1,\varphi_2\rangle+\frac{1}{4}\langle\varphi_1,\varphi_1\rangle \end{pmatrix}$$

is the solution of evolution equation hierarchy

$$u_{tm}=\begin{pmatrix} r_{tm} \\ q_{tm} \end{pmatrix}=JG_m,\ m=0,1,2,\cdots$$

IV. The Hamilton Equation and Its Complete Integrability under the Neumann Constraint

Suppose that $\lambda_1<\lambda_2<\cdots\lambda_N$ is N different eigenvalues of Equation (2.5). $\varphi_{1j},\varphi_{2j}$ are characteristic function[12] of λ_j ($j=1,2,\cdots,N$), so

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}_x = M(u,\lambda_j)\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix} \quad (18)$$

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}_{t_m} = w_m(u,\lambda_j)\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}, \quad (19)$$

$$\Lambda=diag(\lambda_1,\lambda_2,..\lambda_N),$$

$$\varphi_{1j}=(\varphi_{11},...,\varphi_{1N})^T,$$

$$\varphi_{2j}=(\varphi_{21},...\varphi_{2N})^T$$

$Kgrad\lambda=\lambda Jgrad\lambda$, we can get the result:

$$K\begin{pmatrix} \langle\Lambda^k\varphi_1,\varphi_1\rangle \\ \langle\Lambda^k\varphi_1,\varphi_2\rangle \end{pmatrix}=J\begin{pmatrix} \langle\Lambda^{k+1}\varphi_1,\varphi_1\rangle \\ \langle\Lambda^{k+1}\varphi_1,\varphi_2\rangle \end{pmatrix} \quad (20)$$

From (14) and (19), we have

$$G_j=\begin{pmatrix} \langle\Lambda^{j+1}\varphi_1,\varphi_1\rangle \\ \langle\Lambda^{j+1}\varphi_1,\varphi_2\rangle \end{pmatrix} \quad j=-1,0,1,2,\cdots$$

$\Lambda=diag(\lambda_1,...\lambda_N)$;

At the same time,

$$\begin{cases} a_{j-1} \\ b_{j-1} \end{cases}=\begin{pmatrix} \langle\Lambda^j\varphi_1,\varphi_1\rangle \\ \langle\Lambda^j\varphi_1,\varphi_2\rangle \end{pmatrix},\ j=1,2,\cdots \quad (21)$$

$$G_{-1}=\begin{pmatrix} 4 \\ 0 \end{pmatrix}=\begin{pmatrix} \langle\varphi_1,\varphi_1\rangle \\ 2\langle\varphi_1,\varphi_2\rangle \end{pmatrix}$$

Neumann constraint condition [13, 14] is :

$$\Gamma: \begin{cases} \langle\varphi_1,\varphi_1\rangle=4 \\ \langle\varphi_1,\varphi_2\rangle=0 \end{cases} \quad (22)$$

Then $\begin{cases} q=\frac{1}{4}\langle\Lambda\varphi_1,\varphi_1\rangle+\frac{1}{4}\langle\Lambda\varphi_1,\varphi_2\rangle \\ r=\frac{1}{4}\langle\Lambda\varphi_1,\varphi_1\rangle+\frac{1}{4}\langle\Lambda\varphi_2,\varphi_2\rangle \end{cases}$

The Poisson bracket [15] of Smooth Function H and F [16] in symplectic space [17] $(R^{2N},\sum_{k=1}^{N} d\varphi_{2k}\Lambda d\varphi_{1k})$ is defined as followed:

$$\{H,F\}=\sum_{k=1}^{N}\left(\frac{\partial H}{\partial\varphi_{2k}}\frac{\partial F}{\partial\varphi_{1k}}-\frac{\partial H}{\partial\varphi_{1k}}\frac{\partial F}{\partial\varphi_{2k}}\right)$$

Proposition 4.1: (18) and (19) can be written as a finite-dimensional system:

$$\begin{cases} \varphi_{1x}=\frac{\partial H}{\partial\varphi_2} \quad \varphi_{2x}=-\frac{\partial H}{\partial\varphi_1} \\ \varphi_{1t_m}=\frac{\partial H_m}{\partial\varphi_2} \quad \varphi_{2t_m}=-\frac{\partial H_m}{\partial\varphi_1} \end{cases} \quad (23)$$

. H and H_m are *Hamilton* functions here

$$H=-\frac{1}{4}\langle\Lambda\varphi_1,\varphi_1\rangle\langle\varphi_1,\varphi_2\rangle-\frac{1}{4}\langle\Lambda\varphi_1,\varphi_2\rangle\langle\varphi_1,\varphi_2\rangle$$

$$+\langle\Lambda\varphi_1,\varphi_2\rangle+\frac{1}{8}\langle\Lambda\varphi_1,\varphi_1\rangle\langle\varphi_1,\varphi_1\rangle-\langle\Lambda\varphi_1,\varphi_1\rangle+\frac{1}{8}\langle\Lambda\varphi_2,\varphi_2\rangle\langle\varphi_1,\varphi_1\rangle \quad (24)$$

$$H_m=-\frac{1}{2}\langle\Lambda^{m+1}\varphi_1,\varphi_1\rangle\langle\varphi_2,\varphi_2\rangle-4\langle\Lambda^{m+1}\varphi_1,\varphi_1\rangle+\langle\Lambda^{m+1}\varphi_1,\varphi_1\rangle\langle\varphi_1,\varphi_2\rangle+\frac{1}{2}\langle\Lambda^{m+1}\varphi_1,\varphi_1\rangle\langle\varphi_1,\varphi_1\rangle+\frac{1}{2}\sum_{j=0}^{m+1}\langle\Lambda^{j}\varphi_2,\varphi_2\rangle\langle\Lambda^{m-j+1}\varphi_1,\varphi_1\rangle-\frac{1}{2}\sum_{j=0}^{m+1}\langle\Lambda^{j}\varphi_2,\varphi_1\rangle\langle\Lambda^{m-j+1}\varphi_2,\varphi_1\rangle \quad (25)$$

The generator [18] E_k is as followed:

$$E_k=\frac{1}{2}\lambda_k\Gamma_k-\varphi_{1k}{}^2\langle\varphi_1,\varphi_2\rangle+4\varphi_{1k}\varphi_{2k}-4\varphi_{1k}{}^2+\frac{1}{2}\varphi_{1k}{}^2\langle\varphi_1,\varphi_1\rangle-\frac{1}{2}\varphi_{1k}{}^2\langle\varphi_2,\varphi_2\rangle$$

$$(\text{Let }\Gamma_k=\sum_{\substack{j=1,\\ j\neq k}}^{N}\frac{(\varphi_{1k}\varphi_{2j}-\varphi_{2k}\varphi_{1j})^2}{\lambda_k-\lambda_j}),$$

Proposition4.2: (1) $\{E_k, k=1,2,...N\}$ is involutive system [19]: so that

$$\{E_k,E_j\}=0,\forall k,\ j=1,2,...N,$$

and $\{dE_k\}$ has nothing relation with the gradient[20].

(2) $H_\lambda=\sum_{k=1}^{n}\frac{E_k}{\lambda-\lambda_k}=\sum_{m=0}^{\infty}\lambda^{-m-1}G_m$

$$G_m=\sum_{k=1}^{n}\lambda_k{}^m\frac{E_k}{\lambda-\lambda_k}$$

$$H_m=G_{m+1}$$

Among them, $H_m=\sum_{k=1}^{n}\lambda_k{}^{m+1}E_k, m=1,2,\cdots$.

Proof (1): From the define of Poisson bracket,

$$\langle\varphi_{1k},\varphi_{1n}\rangle=\langle\varphi_{2k},\varphi_{2n}\rangle=0$$

$$\langle\varphi_{1k},\varphi_{2n}\rangle=\delta_{kn}=\begin{cases}1 & k=n\\ 0 & k\neq n\end{cases}$$

$$(\langle\varphi_1,\varphi_1\rangle,B_{kn})=(\langle\varphi_2,\varphi_2\rangle,B_{kn})=(\langle\varphi_1,\varphi_2\rangle,B_{kn})=0$$

$$(\Gamma_k,\Gamma_n)=\left(\sum_{\substack{j=1,\\ j\neq k}}^{N}\frac{(\varphi_{1k}\varphi_{2j}-\varphi_{2k}\varphi_{1j})^2}{\lambda_k-\lambda_j},\sum_{\substack{j=1,\\ j\neq n}}^{N}\frac{(\varphi_{1n}\varphi_{2j}-\varphi_{2n}\varphi_{1j})^2}{\lambda_n-\lambda_j}\right)=0$$

$$(\varphi_{1k}^2,\Gamma_n)=(\varphi_{1k}^2,\sum_{\substack{j=1,\\ j\neq n}}^{N}\frac{(\varphi_{1n}\varphi_{2j}-\varphi_{2n}\varphi_{1j})^2}{\lambda_n-\lambda_j})=4(\lambda_k-\lambda_n)^{-1}\varphi_{1k}\varphi_{1n}B_{nk}$$

$$(\varphi_{2k}^2,\Gamma_n)=(\varphi_{2k}^2,\sum_{\substack{j=1,\\ j\neq n}}^{N}\frac{(\varphi_{1n}\varphi_{2j}-\varphi_{2n}\varphi_{1j})^2}{\lambda_n-\lambda_j})=4(\lambda_k-\lambda_n)^{-1}\varphi_{2k}\varphi_{2n}B_{nk}$$

$$(\varphi_{1k}\varphi_{2k},\Gamma_n)=(\varphi_{1k}\varphi_{2k},\sum_{\substack{j=1,\\ j\neq n}}^{N}\frac{(\varphi_{1n}\varphi_{2j}-\varphi_{2n}\varphi_{1j})^2}{\lambda_n-\lambda_j})=2(\lambda_k-\lambda_n)^{-1}(\varphi_{2k}\varphi_{1n}+\varphi_{1k}\varphi_{2n})B_{nk}$$

Therefore, $(E_k,E_n)=0$

Proof (2):

$$\sum_{k=1}^{n}\frac{E_k}{\lambda-\lambda_k}=\sum_{k=1}^{n}\frac{1}{\lambda}\frac{E_k}{1-\frac{\lambda_k}{\lambda}}E_k=\sum_{k=1}^{n}\frac{1}{\lambda}\sum_{m=0}^{\infty}\left(\frac{\lambda_k}{\lambda}\right)^m E_k=\sum_{m=0}^{\infty}\lambda^{-m-1}\sum_{k=1}^{n}(\lambda_k)^m E_k=\sum_{m=0}^{\infty}\lambda^{-m-1}G_m$$

Proposition 4.3: let $f_1=\langle\varphi_1,\varphi_1\rangle-4$, $f_2=\langle\varphi_1,\varphi_2\rangle$, then the solutions as follows on Γ:

(1) $\{H, f_j\}=0,\ j=1,2$

(2) $\{f_j,E_k\}=0,\ j=1,2\quad k=1,2,\cdots n$

(3) $\det\{f_i, f_j\}\neq 0,\quad i,j=1,2$

(4) $\{H,E_k\}=0, k=1,2,...N.$

(5) $\{H,H_m\}=0, m=0,1,2,...$

(6) $\{H_n,H_m\}=0, m,n=0,1,2,...$

(7) $(R^{2n},\sum_{k=1}^{n}d\varphi_{1k}\Lambda d\varphi_{1k},H)$,

$(R^{2n},\sum_{k=1}^{n}d\varphi_{1k}\Lambda d\varphi_{1k},H)$ is the completely integrable system in the Liouville sense [21].

Proof: $\{H, f_j\}=0,\ j=1,2$

$\{f_j,E_k\}=0,\ j=1,2\quad k=1,2,\cdots n$

$\det\{f_i, f_j\}\neq 0,\quad i,j=1,2$; $\{H,E_k\}=0, k=1,2,...N.$

can be proved by calculation. $\{H_n,H_m\}=0, m,n=0,1,2,...$ can be proved by the previous proposition. $(R^{2n},\sum_{k=1}^{n}d\varphi_{1k}\Lambda d\varphi_{1k},H)$, $(R^{2n},\sum_{k=1}^{n}d\varphi_{1k}\Lambda d\varphi_{1k},H)$ are proved by Liouville theorem. Hence the proposition is concluded.

Proposition 4.4: if φ_1,φ_2 are involutive solutions of the Hamilton regular system, so $\begin{pmatrix}r\\ q\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\langle\varphi_1,\varphi_2\rangle-\frac{1}{2}\langle\varphi_1,\varphi_1\rangle\\ \frac{1}{4}\langle\varphi_1,\varphi_2\rangle+\frac{1}{4}\langle\varphi_1,\varphi_1\rangle\end{pmatrix}$ is the solution of evolution equation hierarchy $\begin{pmatrix}r\\ q\end{pmatrix}_{t_m}=KG_m, m=0,1,2,\cdots$.

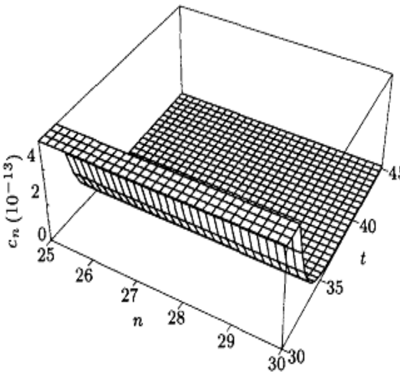

Figure6 Solition solution

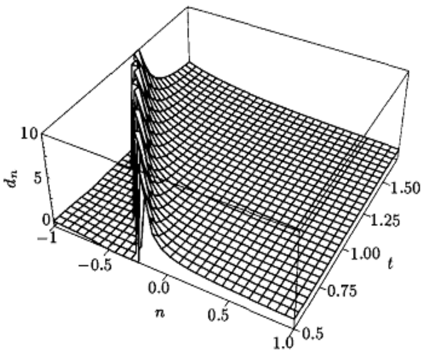

Figure7 Solition solution

V. CONCLUSIONS

The results of this paper have been algebraic in nature. We have considered the second order matrix eigenvalue problem (1). In the first part of section 2, we gave Lax representation and evolution equation hierarchy which related to eigenvalue problem. In section 3, Hamilton equation and its complete integrability under the Bargmann constraint are given. In section 4, Hamilton equation and its complete integrability under the Neumann constraint are given.

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