Nonlinear Evolution Equations for Second-order Spectral Problem

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Abstract—Soliton equations are infinite-dimensional integrable systems described by nonlinear evolution equations. As one of the soliton equations, long wave equation takes on profound significance of theory and reality. By using the method of nonlinearization, the relation between long wave equation and second-order eigenvalue problem is generated. Based on the nonlinearized Lax pairs, Euler-Lagrange function and Legendre transformations, a reasonable Jacobi-Ostrogradsky coordinate system is obtained. Moreover, by means of the Bargmann constrained condition between the potential function and the eigenfunction, the Lax pairs is equivalent to matrix spectral problem. Furthermore, the involutive representations of the solutions for long wave equation are generated.

Index Terms—spectral problem, Hamilton canonical system, Bargmann constraint, integrable system, involutive solution

I. INTRODUCTION

In 1895, Korteweg and de Vries [1-2] derived a nonlinear evolution equation as follows:

$$\frac{\partial \eta}{\partial \tau} = \frac{3}{2} \sqrt{\frac{g}{h}} \frac{\partial}{\partial \xi} (\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma \frac{\partial^2 \eta}{\partial \xi^2})$$

$$\sigma = \frac{1}{3}h^3 - \frac{Th}{\rho g}$$

By making the transformation

$$t = \frac{1}{2}\sqrt{\frac{g}{h\sigma}}\tau , \ x = -\sigma^{1/2}\xi , \ u = \frac{1}{2}\eta + \frac{1}{3}\alpha$$

the famous KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

is obtained. It aroused an increasing interest among scientific researchers in the field of mathematics and physics, so more and more scientists have been interested in searching various methods to obtain solutions of some partial differential equation. Many effective methods have been proposed, for example, Hirota method, the inverse scattering, Darboux transformation, Painlevé expansion, Bäcklund transformation, algebraic method and so on [1-7]. Using the inverse scattering method, we could obtain the N-soliton solution of KdV equation (see Fig. 1 and Fig. 2).
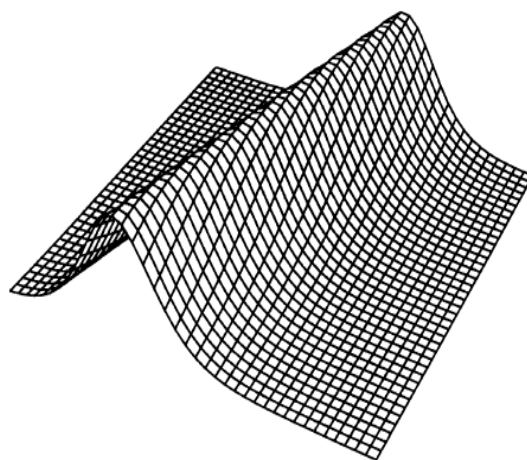

Figure1. One-soliton solution of the KdV equation

In our paper, by nonlinearization [8-15] of spectral problems, we considered the spectral

$$L\varphi = (\partial^2 + \partial q + p)\varphi = \lambda\varphi_x$$

The paper is structured as follows. In Sect.2, the adjoint Lax pairs of the spectral problem is generalized. In Sect.3, based on the Euler-Lagrange equations and Legendre transformations, a suitable Jacobi-Ostrogradsky coordinate te is been found. Section 4 and Sect.5 are devoted to establishing the Liouville integrability of the resulting Hamiltonian systems from the 2nd-order spectral problems.
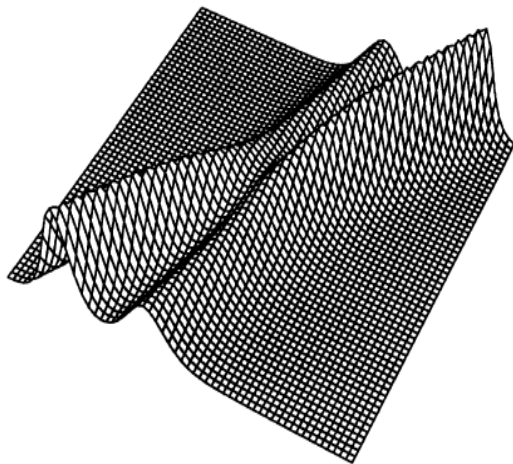
Figure2. Two-soliton solution of the KdV equation

II. LAX PAIRS AND EVOLUTION EQUATIONS

Let us take the 2nd-order problem

$$L\varphi = (\partial^2 + \partial q + p)\varphi = \lambda\varphi_x \tag{1}$$

where $q = q(x,t) \in R, p = p(x,t) \in R$, $q, p \neq const$ are potential functions, λ is a complex eigenparameter, $\partial = \partial/\partial x$, $\partial\partial^{-1} = \partial^{-1}\partial = 1$.

Suppose Ω is the basic interval of (1), for the sake of simplicity, we assume that if the potentials q, p, φ and all their derivatives with respect to x tend to zero, then $\Omega = (-\infty, +\infty)$; If they are all periodic T functions, then $\Omega = [0, 2T]$.

Definition 2.1 Assume that our linear space is equipped with a L_2 scalar product $(\cdot, \cdot)_{L_2(\Omega)}$:

$$(\varphi, h)_{L_2(\Omega)} = \int_\Omega \varphi h^* dx < \infty$$

symbol * denotes the complex conjugate.

Definition 2.2 Operator $\bar{A}$ is an adjoint operator of A , if $(A\varphi, h)_{L_2(\Omega)} = (\varphi, \bar{A}h)_{L_2(\Omega)}$.

Using definition 2.2, we get

$$\bar{L}\psi = (\partial^2 - q\partial + p)\psi = -\lambda^*\psi_x \tag{2}$$

In order to derive the evolution equation related to the spectral problem (1), we consider the stationary zero curvature equation

$$\lambda\omega_x + [\omega, L] = \lambda\omega_x + \omega L - L\omega \tag{3}$$

Take

$$\omega = \sum_{j=0}^{\infty}[-a_{j-1} - b_{j-1x} + b_{j-1}\partial]\lambda^{-j}$$

and set

$$G_{-1} = \begin{pmatrix} a_{-1} \\ b_{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4}$$

therefore, we obtain the recursive relation

$$KG_{j-1} = JG_j, j = 0, 1, 2\ldots, \tag{5}$$

where

$$J = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \tag{6}$$

$$K = \begin{pmatrix} 2\partial & \partial^2 + \partial q \\ q\partial - \partial^2 & p\partial + \partial p \end{pmatrix} \tag{7}$$

$$G_j = \begin{pmatrix} a_j \\ b_j \end{pmatrix}$$

TABLE I.
THE LENARD SEQUENCE

G_j	a_j	b_j
$j = -1$	0	1
$j = 0$	p	q
$j = 1$	$-p_x + 2qp$	$2p + q_x + q^2$
…		

Now, we consider the auxiliary spectral problem

$$\varphi_{t_m} = \omega_m\varphi \quad m = 0, 1, 2,\ldots \tag{8}$$

with

$$\omega_m = \sum_{j=0}^{m}[-a_{j-1} - b_{j-1x} + b_{j-1}\partial]\lambda^{m-j} \tag{9}$$

Then, the isospectral $(i.e. \lambda_{t_m} = 0)$ compatible condition

$$L_{t_m} = \lambda\omega_{mx} + [\omega_m, L] = \lambda\omega_{mx} + \omega_m L - L\omega_m \tag{10}$$

of the Lax pairs

$$\begin{cases} L\varphi = \lambda\varphi_x \\ \varphi_{t_m} = \omega_m\varphi \quad m = 0, 1, 2,\ldots \end{cases} \tag{11}$$

determines a $(m+1)$ -order long-wave equation

$$\begin{pmatrix} q_{t_m} \\ p_{t_m} \end{pmatrix} = JG_m = KG_{m-1} \quad m = 0, 1, 2,\ldots \tag{12}$$

For example, when $m = 1$ and $m = 2$, we can get the first and the second nonlinear systems, the results are shown in Table Ⅱ. When $m = 1$, it is the long-wave equation. When $m = 2$ and $q \equiv 0$, it is exactly the famous KdV equations.

TABLE II.
THE FIRST AND THE SECOND NONLINEAR SYSTEM

Evolution equation	$m = 1$	$m = 2$ and $q \equiv 0$
q_{t_m}	$2p_x + q_{xx} + 2qq_x$	0
p_{t_m}	$-p_{xx} + 2(qp)_x$	$p_{t_2} = p_{xxx} + 3(p^2)_x$

In order to give the constraints between the potentials and the eigenfunctions, it is necessary to calculate the functional gradient with respect to the potential functions.

Proposition 2.3: [11]

i) If λ is an eigenvalue of (1), then λ is real.

ii) If φ is an eigenfunction of (1) and ψ is an eigenfunction of (2), then φ and ψ can be taken real functions.

iii) If φ is an eigenfunction corresponding to the eigenvalue λ of (1) and ψ is an eigenfunction corresponding to the eigenvalue λ of (2), then

$$\nabla \lambda = \begin{pmatrix} \dfrac{\delta \lambda}{\delta q} \\ \dfrac{\delta \lambda}{\delta p} \end{pmatrix} = (\int_\Omega \varphi_x \psi dx)^{-1} \begin{pmatrix} -\varphi \psi_x \\ \varphi \psi \end{pmatrix}$$

and

$$K \nabla \lambda = \lambda J \nabla \lambda \tag{13}$$

Proof: In fact, from (1) and (2), we have

$$\lambda \int_\Omega \varphi_x \psi dx = \int_\Omega L\varphi(\psi^*)^* dx$$
$$= \int_\Omega \varphi(\bar{L}\psi^*)^* dx$$
$$= \int_\Omega \varphi(\bar{L}^*\psi^*)^* dx$$
$$= -\lambda^* \int_\Omega \varphi \psi_x dx$$
$$= \lambda^* \int_\Omega \varphi_x \psi dx$$

so

$$\lambda = \lambda^*$$

If φ is a complex eigenfunction of (1) on λ, and $\varphi = a + ib$, a, b are real functions, from $L\varphi = \lambda \varphi_x$ and λ is real, then

$$\begin{cases} La = \lambda a_x \\ Lb = \lambda b_x \end{cases}$$

so a, b are eigenfunction of (1) on λ, φ can be taken real function. Similarly, ψ can be taken real function.

Let

$$h^\square = \dfrac{d}{d\varepsilon}\bigg|_{\varepsilon=0} h(\lambda + \varepsilon \delta \lambda, q + \varepsilon \delta q, p + \varepsilon \delta p),$$

by

$$L\varphi = \lambda \varphi_x$$
$$(L\varphi)^\square = (\lambda \varphi_x)^\square$$
$$L^\square \varphi + L\varphi^\square = \lambda^\square \varphi_x + \lambda \varphi_x^\square$$

and

$$\int_\Omega (L\varphi^\square)\psi dx = \int_\Omega \varphi^\square(\bar{L}\psi)dx$$
$$= \int_\Omega \varphi^\square(-\lambda \psi_x)dx$$
$$= \int_\Omega \lambda \varphi_x^\square \psi dx$$

so

$$\int_\Omega \lambda^\square \varphi_x \psi dx = \int_\Omega L^\square \varphi \psi dx$$
$$= \int_\Omega ((\delta q \varphi)_x + \delta p \varphi)\psi dx$$
$$= \int_\Omega (-\delta q \varphi \psi_x + \delta p \varphi \psi)dx$$

then

$$\nabla \lambda = \begin{pmatrix} \dfrac{\delta \lambda}{\delta q} \\ \dfrac{\delta \lambda}{\delta p} \end{pmatrix} = (\int_\Omega \varphi_x \psi dx)^{-1} \begin{pmatrix} -\varphi \psi_x \\ \varphi \psi \end{pmatrix}$$

by (6) and (7), (13) holds. $\square$

III. BARGMANN SYSTEMS AND THE HAMILTON CANONICAL FORMS

We suppose $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ are the eigenvalues of the eigenvalue problem (1) and (2), φ_j, ψ_j are the eigenfunction for λ_j ($j = 1, 2, \cdots, N$). Let

$$\Lambda = diag(\lambda_1, \lambda_2, \cdots \lambda_N),$$
$$\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_N)^T,$$
$$\Psi = (\psi_1, \psi_2, \ldots, \psi_N)^T$$

Now, we consider the Bargmann constraint [16-18]

$$\begin{cases} q = <\Phi, \Psi> \\ p = -<\Phi, \Psi_x> \end{cases} \tag{14}$$

here

$$<\Phi, \Psi> = \sum_{k=1}^N \varphi_j \psi_j$$

namely

$$G_0 = \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -<\Phi, \Psi_x> \\ <\Phi, \Psi> \end{pmatrix}$$

$$G_j = \begin{pmatrix} <-\Lambda^j \Phi, \Psi_x> \\ <\Lambda^j \Phi, \Psi> \end{pmatrix} \quad j = 0, 1, 2 \cdots \tag{15}$$

so the eigenvalue problem (1) and (2) are equivalent to the systems

$$\begin{cases} \Phi_{xx} + (<\Phi, \Psi>\Phi)_x - <\Phi, \Psi_x>\Phi = \Lambda\Phi_x \\ \Psi_{xx} - <\Phi, \Psi>\Psi_x - <\Phi, \Psi_x>\Psi = -\Lambda\Psi_x \end{cases} \tag{16}$$

and (16) are called the Bargmann systems for the eigenvalue problem (1) and (2).

Let

$$\hat{I} = \int_\Omega I dx \tag{17}$$

where the Lagrange function I is defined as follows:

$$I = <\Phi_x, \Psi_x> + \frac{1}{2}<\Phi, \Psi><\Phi, \Psi_x> + \frac{1}{2}<\Lambda\Phi_x, \Psi>$$
$$-\frac{1}{2}<\Lambda\Phi, \Psi_x> - \frac{1}{2}<\Phi, \Psi><\Phi_x, \Psi>$$

Proposition 3.1: The Bargmann systems (16) of the eigenvalue problem (1) and (2) are equivalent to the Euler-Lagrange equation systems:

$$\begin{cases} \dfrac{\delta \hat{I}}{\delta \Phi} = 0 \\ \dfrac{\delta \hat{I}}{\delta \Psi} = 0 \end{cases} \tag{18}$$

Proof: By (17), we have

$$\frac{\delta \hat{I}}{\delta \varphi_j} = \psi_{jxx} - <\Phi, \Psi_x>\psi_j - <\Phi, \Psi>\psi_{jx} + \lambda \psi_{jx}$$

$$= <\Phi, \Psi_x>\psi_j + <\Phi, \Psi>\psi_{jx} - \lambda \psi_{jx}$$

$$-<\Phi,\Psi_x>\psi_j-<\Phi,\Psi>\psi_{jx}+\lambda\psi_{jx}$$
$$=0$$

similarly,

$$\frac{\delta\hat{I}}{\delta\psi_j}=-\varphi_{jxx}-(<\Phi,\Psi>\varphi_j)_x+<\Phi,\Psi_x>\varphi_{jx}+\lambda\varphi_{jx}$$

$$=(<\Phi,\Psi>\varphi_j)_x-<\Phi,\Psi_x>\varphi_{jx}-\lambda\varphi_{jx}$$

$$-(<\Phi,\Psi>\varphi_j)_x+<\Phi,\Psi_x>\varphi_{jx}+\lambda\varphi_{jx}$$

$$=0 \qquad \Box$$

Now, the Poisson bracket [19] of the real-valued functions F and H in the symplectic [20] space ($w=\sum_{j=1}^{2}dz_j\wedge dy_j, R^{4N}$) is defined as follows:

$$\{F,H\}=\sum_{j=1}^{2}\sum_{k=1}^{N}(\frac{\partial F}{\partial y_{jk}}\frac{\partial H}{\partial z_{jk}}-\frac{\partial F}{\partial z_{jk}}\frac{\partial H}{\partial y_{jk}})$$

$$=\sum_{j=1}^{2}(<F_{yj},H_{zj}>-<F_{zj},H_{yj}>)$$

Based on the the Euler-Lagrange equation (18), the Jacobi-Ostrogradsky coordinates can be found, and the Bargmann systems (16) can be written in the Hamilton canonical equation systems [21].

Let

$$u_1=\Phi,\quad u_2=\Psi,\quad g=\sum_{j=1}^{2}<u_{jx},v_j>-I$$

Our aim is to find that the coordinates $\{v_1,v_2\}$ and g satisfy the following Hamilton canonical equations:

$$\begin{cases}u_{jx}=\{u_j,g\}=\dfrac{\partial g}{\partial v_j}\\[2mm]v_{jx}=\{v_j,g\}=-\dfrac{\partial g}{\partial u_j}\end{cases}\quad j=1,2$$

By directly computing, we have

$$\begin{cases}v_1=\Psi_x-\dfrac{1}{2}q\Psi+\dfrac{1}{2}\Lambda\Psi\\[2mm]v_2=\Phi_x+\dfrac{1}{2}q\Phi-\dfrac{1}{2}\Lambda\Phi\\[2mm]v_{1x}=\dfrac{1}{2}<\Phi,\Psi_x>\Psi+\dfrac{1}{2}<\Phi,\Psi>\Psi_x\\[2mm]\qquad -\dfrac{1}{2}<\Phi_x,\Psi>\Psi-\dfrac{1}{2}\Lambda\Psi_x\\[2mm]v_{2x}=\dfrac{1}{2}<\Phi,\Psi_x>\Phi-\dfrac{1}{2}<\Phi_x,\Psi>\Phi\\[2mm]\qquad -\dfrac{1}{2}<\Phi,\Psi>\Phi_x+\dfrac{1}{2}\Lambda\Phi_x\end{cases}$$

So, if we take the Jacobi-Ostrogradsky coordinates as follows:

$$\begin{cases}u_1=\Phi\\u_2=\Psi\\v_1=\Psi_x-\dfrac{1}{2}q\Psi+\dfrac{1}{2}\Lambda\Psi\\[2mm]v_2=\Phi_x+\dfrac{1}{2}q\Phi-\dfrac{1}{2}\Lambda\Phi\end{cases}\qquad(19)$$

Then the Bargmann systems (16) are equivalent to the following Hamilton canonical systems:

$$\begin{cases}u_{jx}=\dfrac{\partial g}{\partial v_j}\\[2mm]v_{jx}=-\dfrac{\partial g}{\partial u_j}\end{cases}\quad j=1,2$$

where

$$g=<v_1,v_2>+\frac{1}{2}<u_1,u_2><u_2,v_2>-\frac{1}{2}<\lambda u_2,v_2>$$

$$-\frac{1}{2}<u_1,u_2><u_1,v_1>-\frac{1}{4}<u_1,u_2>^3+\frac{1}{2}<\lambda u_1,v_1>$$

$$+\frac{1}{2}<u_1,u_2><\lambda u_1,u_2>-\frac{1}{4}<\lambda^2 u_1,u_2>$$

By (19), the Jacobi-Ostrogradsky coordinates can be written in the following form:

$$\begin{cases}y_1=\Phi,\\[2mm]y_2=\Phi_x+\dfrac{1}{2}q\Phi-\dfrac{1}{2}\Lambda\Phi\\[2mm]z_2=\Psi\\[2mm]z_1=-\Psi_x+\dfrac{1}{2}q\Psi-\dfrac{1}{2}\Lambda\Psi\end{cases}\qquad(20)$$

then, we have:

Theorem 3.2: The Bargmann systems (16) of the eigenvalue problem (1) and (2) are equivalent to the following Hamilton canonical systems:

$$\begin{cases}y_{jx}=\dfrac{\partial h}{\partial z_j}\\[2mm]z_{jx}=-\dfrac{\partial h}{\partial y_j}\end{cases}\quad j=1,2$$

where

$$h=<y_2,z_1>-\frac{1}{2}<y_1,z_2><y_2,z_2>+\frac{1}{2}<\Lambda y_2,z_2>$$

$$-\frac{1}{2}<y_1,z_2><y_1,z_1>+\frac{1}{4}<y_1,z_2>^3+\frac{1}{2}<\Lambda y_1,z_1>$$

$$-\frac{1}{2}<\Lambda y_1,z_2><y_1,z_2>+\frac{1}{4}<\Lambda^2 y_1,z_2>$$

and $h=-g$.

IV. NONLINEARIZATION OF THE LAX PAIRS

Form (20) and Theorem 3.2, the Bargmann systems (16) have the equivalence form

$$\begin{cases}Y_x=MY\\Z_x=-M^T Z\end{cases}$$

where

$$Y=\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}\Phi\\\Phi_x+\dfrac{1}{2}q\Phi-\dfrac{1}{2}\Lambda\Phi\end{pmatrix}$$

$$Z=\begin{pmatrix}z_1\\z_2\end{pmatrix}=\begin{pmatrix}-\Psi_x+\dfrac{1}{2}q\Psi-\dfrac{1}{2}\Lambda\Psi\\\Psi\end{pmatrix}$$

$$M = \begin{pmatrix} M_{11} & E \\ M_{21} & M_{22} \end{pmatrix}$$

where

$$M_{11} = -\frac{1}{2}qE + \frac{1}{2}\Lambda$$

$$M_{21} = \frac{1}{4}q^2E - pE - \frac{1}{2}q_xE - \frac{1}{2}\Lambda q + \frac{1}{4}\Lambda^2$$

$$M_{22} = -\frac{1}{2}qE + \frac{1}{2}\Lambda$$

$$E = E_{N \times N} = diag(1,1,\ldots,1)$$

Proposition 4.1: The Lax pairs (11) for the $(m+1)$-order evolution equation (12) is equivalent to

$$\begin{cases} Y_x = MY, & Z_x = -M^TZ; \\ Y_{t_m} = W_mY, & Z_{t_m} = -W_m^TZ, \quad m = 0,1,2\ldots, \end{cases} \quad (21)$$

where

$$W_m = \sum_{j=0}^{m} \begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix} \Lambda^{m-j}$$

$$A_m = -a_{j-1} - \frac{1}{2}qb_{j-1} - b_{j-1x} + \frac{1}{2}\Lambda b_{j-1}$$

$$B_m = b_{j-1}$$

$$C_m = -\frac{1}{2}b_{j-1xx} - \frac{1}{2}q_xb_{j-1} - pb_{j-1} + \frac{1}{4}q^2b_{j-1} - \frac{1}{2}\Lambda qb_{j-1}$$
$$+ \frac{1}{4}\Lambda^2 b_{j-1}$$

$$D_m = -a_{j-1} - \frac{1}{2}qb_{j-1} + \frac{1}{2}\Lambda b_{j-1}$$

By (14) and (20), we have the Bargmann constraint

$$\begin{cases} q = <y_1, z_2> \\ p = <y_1, z_1> - \frac{1}{2}<y_1, z_2>^2 + \frac{1}{2}<\Lambda y_1, z_2> \end{cases}$$

$$(22)$$

$$G_j = \begin{pmatrix} a_j \\ b_j \end{pmatrix} = \begin{pmatrix} G_{11} \\ <\Lambda^j y_1, z_2> \end{pmatrix}, j = 0,1,2,\ldots,$$

$$(23)$$

where

$$G_{11} = <\Lambda^j y_1, z_1> - \frac{1}{2}<y_1, z_2><\Lambda^j y_1, z_2> + \frac{1}{2}<\Lambda^{j+1} y_1, z_2>$$

Substituting the Bargmann constraint (22) and (23) into (21), the Lax pairs (24) for the $(m+1)$-order evolution equation (12) is equivalent to the following forms:

$$\begin{cases} Y_x = \bar{M}Y, & Z_x = -\bar{M}^TZ \\ Y_{t_m} = \bar{W}_mY & Z_{t_m} = -\bar{W}_m^TZ \quad m = 0,1,2,\ldots \end{cases} \quad (24)$$

where

$$\bar{M} = \begin{pmatrix} M_{11} & E \\ M_{21} & M_{22} \end{pmatrix}$$

$$\bar{W}_m = \begin{pmatrix} \bar{A}_m & \bar{B}_m \\ \bar{C}_m & \bar{D}_m \end{pmatrix}$$

$$M_{11} = -\frac{1}{2}<y_1, z_2> + \frac{1}{2}\Lambda$$

$$M_{21} = \frac{1}{4}\Lambda^2 + \frac{3}{4}<y_1, z_2>^2 - \frac{1}{2}<y_1, z_1> - \frac{1}{2}<y_2, z_2>$$
$$-\frac{1}{2}<\Lambda y_1, z_2> - \frac{1}{2}\Lambda<y_1, z_2>$$

$$M_{22} = -\frac{1}{2}<y_1, z_2> + \frac{1}{2}\Lambda$$

$$\bar{A}_m = \sum_{j=1}^{m}[-<\Lambda^{j-1} y_2, z_2>]\Lambda^{m-j} + \frac{1}{2}\Lambda^{m+1} - \frac{1}{2}<\Lambda^m y_1, z_2>$$

$$\bar{B}_m = \sum_{j=1}^{m}[<\Lambda^{j-1} y_1, z_2>]\Lambda^{m-j} + \Lambda^m$$

$$\bar{C}_m = \sum_{j=1}^{m}[<\Lambda^{j-1} y_2, z_1>]\Lambda^{m-j} - \frac{1}{2}(<y_1, z_1> + <y_2, z_2>)\Lambda^m$$
$$-\frac{1}{4}<y_1, z_2>\Lambda^{m+1} - \frac{1}{4}<\Lambda y_1, z_2>\Lambda^m + \frac{1}{4}\Lambda^{m+2}$$
$$-\frac{1}{4}<\Lambda^{m+1} y_1, z_2> - \frac{1}{4}<\Lambda^m y_1, z_2>\Lambda$$
$$+\frac{1}{2}<y_1, z_2><\Lambda^m y_1, z_2> + \frac{1}{4}<y_1, z_2>^2\Lambda^m$$

$$\bar{D}_m = \sum_{j=1}^{m}[-<\Lambda^{j-1} y_1, z_1>]\Lambda^{m-j} + \frac{1}{2}\Lambda^{m+1} - \frac{1}{2}<\Lambda^m y_1, z_2>$$

Theorem 4.2: On the Bargmann constraint (22), the nonlinearized Lax pairs (24) for the $(m+1)$-order long wave equation (12) can be written as the following Hamilton systems [11-12]:

$$\begin{cases} Y_x = \frac{\partial h}{\partial Z}, & Z_x = -\frac{\partial h}{\partial Y}; \\ Y_{t_m} = \frac{\partial h_m}{\partial Z}, & Z_{t_m} = -\frac{\partial h_m}{\partial Y}, \quad m = 0,1,2,\ldots, \end{cases}$$

$$(25)$$

where

$$h_m = \frac{1}{2}<\Lambda^{m+1} y_1, z_1> + \frac{1}{2}<\Lambda^{m+1} y_2, z_2> + \frac{1}{4}<\Lambda^{m+2} y_1, z_2>$$
$$+\frac{1}{4}<y_1, z_2>^2<\Lambda^m y_1, z_2> - \frac{1}{4}<\Lambda^m y_1, z_2><\Lambda y_1, z_2>$$
$$-\frac{1}{4}<\Lambda^{m+1} y_1, z_2><y_1, z_2> + <\Lambda^m y_2, z_1>$$
$$-\frac{1}{2}<\Lambda^m y_1, z_2>(<y_1, z_1> + <y_2, z_2>)$$
$$+\sum_{j=1}^{m}\begin{vmatrix} <\Lambda^{j-1} y_1, z_2> & <\Lambda^{m-j} y_2, z_2> \\ <\Lambda^{j-1} y_1, z_1> & <\Lambda^{m-j} y_2, z_1> \end{vmatrix}$$

and $h_0 = h$.

Proof: $\frac{\partial h}{\partial z_1} = y_2 - \frac{1}{2}<y_1, z_2>y_1 + \frac{1}{2}\Lambda y_1$

$$= y_{1x}$$

$$\frac{\partial h}{\partial z_2} = \frac{1}{2}\Lambda y_2 - \frac{1}{2}<y_1, z_2>y_2 - \frac{1}{2}<y_2, z_2>y_1$$

$$-\frac{1}{2}<y_1,z_1>y_1+\frac{3}{4}<y_1,z_2>^2 y_1+\frac{1}{4}\Lambda^2 y_1$$

$$-\frac{1}{2}<\Lambda y_1,z_2>y_1-\frac{1}{2}<y_1,z_2>\Lambda y_1$$

$$=y_{2x}$$

$$\frac{\partial h_m}{\partial z_1}=\frac{1}{2}\Lambda^{m+1}y_1-\frac{1}{2}<\Lambda^m y_1,z_2>y_1+\Lambda^m y_2$$

$$+\sum_{j=1}^{m}[<\Lambda^{j-1}y_1,z_2>y_2-<\Lambda^{m-j}y_2,z_2>y_1]\Lambda^{m-j}$$

$$=y_{1t_m}$$

$$\frac{\partial h_m}{\partial z_2}=\frac{1}{2}\Lambda^{m+1}y_2+\frac{1}{4}\Lambda^{m+2}y_1-\frac{1}{2}<y_1,z_1>\Lambda^m y_1$$

$$-\frac{1}{2}<\Lambda^m y_1,z_2>y_2-\frac{1}{2}<y_2,z_2>\Lambda^m y_1$$

$$+\frac{1}{4}<y_1,z_2>^2\Lambda^m y_1+\frac{1}{2}<y_1,z_2><\Lambda^m y_1,z_2>y_1$$

$$-\frac{1}{4}<\Lambda^m y_1,z_2>\Lambda y_1-\frac{1}{4}<\Lambda y_1,z_2>\Lambda^m y_1$$

$$-\frac{1}{4}<\Lambda^{m+1}y_1,z_2>y_1-\frac{1}{4}<y_1,z_2>\Lambda^{m+1}y_1$$

$$+\sum_{j=1}^{m}[<\Lambda^{j-1}y_2,z_1>y_1-<\Lambda^{j-1}y_1,z_1>y_2]\Lambda^{m-j}$$

$$=y_{2t_m}$$

so

$$Y_x=\frac{\partial h}{\partial Z},\ Y_{t_m}=\frac{\partial h_m}{\partial Z}$$

Similarly, we have

$$\frac{\partial h}{\partial y_1}=\frac{1}{2}\Lambda z_1-\frac{1}{2}<y_2,z_2>z_2+\frac{3}{4}<y_1,z_2>^2 z_2$$

$$+\frac{1}{4}\Lambda^2 z_2-\frac{1}{2}<y_1,z_2>z_1-\frac{1}{2}<y_1,z_1>z_2$$

$$-\frac{1}{2}<\Lambda y_1,z_2>z_2-\frac{1}{2}<y_1,z_2>\Lambda z_2$$

$$=-z_{1x}$$

$$\frac{\partial h}{\partial y_2}=z_1-\frac{1}{2}<y_1,z_2>z_2+\frac{1}{2}\Lambda z_2$$

$$=-z_{2x}$$

$$\frac{\partial h_m}{\partial y_1}=\frac{1}{2}\Lambda^{m+1}z_1-\frac{1}{2}<\Lambda^m y_1,z_2>z_1-\frac{1}{2}<y_1,z_1>\Lambda^m z_2$$

$$+\frac{1}{4}<y_1,z_2>^2\Lambda^m z_2+\frac{1}{2}<y_1,z_2><\Lambda^m y_1,z_2>z_2$$

$$-\frac{1}{4}<\Lambda^{m+1}y_1,z_2>z_2-\frac{1}{4}<y_1,z_2>\Lambda^{m+1}z_2$$

$$-\frac{1}{4}<\Lambda^m y_1,z_2>\Lambda z_2-\frac{1}{4}<\Lambda y_1,z_2>\Lambda^m z_2$$

$$-\frac{1}{2}<y_2,z_2>\Lambda^m z_2+\frac{1}{4}\Lambda^{m+2}z_2$$

$$+\sum_{j=1}^{m}[<\Lambda^{j-1}y_2,z_1>z_2-<\Lambda^{j-1}y_2,z_2>z_1]\Lambda^{m-j}$$

$$=-z_{1t_m}$$

$$\frac{\partial h_m}{\partial y_2}=\frac{1}{2}\Lambda^{m+1}z_2+\Lambda^m z_1-\frac{1}{2}<\Lambda^m y_1,z_2>z_2$$

$$+\sum_{j=1}^{m}[<\Lambda^{j-1}y_1,z_2>z_1-<\Lambda^{j-1}y_1,z_1>z_2]\Lambda^{m-j}$$

$$=-z_{2t_m}$$

so

$$Z_x=-\frac{\partial h}{\partial Y},\ Z_{t_m}=-\frac{\partial h_m}{\partial Y}$$

□

V. LIOUVILLE COMPLETELY INTEGRABLE SYSTEMS

Now we discuss the completely integrability for the Bargmann systems (25).

Let

$$\begin{cases} E_k^{(1)}=\frac{1}{2}\lambda_k y_{1k}z_{1k}+\frac{1}{2}\lambda_k y_{2k}z_{2k} \\ E_k^{(2)}=-\frac{1}{2}y_{1k}z_{2k}(<y_1,z_1>+<y_2,z_2>)+y_{2k}z_{1k} \\ \quad -\frac{1}{4}\lambda_k y_{1k}z_{2k}<y_1,z_2>+\frac{1}{4}y_{1k}z_{2k}<y_1,z_2>^2 \\ \quad +\frac{1}{4}\lambda_k^2 y_{1k}z_{2k}-\frac{1}{4}y_{1k}z_{2k}<\Lambda y_1,z_2>-\Gamma_k^{(1,2)} \end{cases}$$

where

$$\Gamma_k^{(1,2)}=\sum_{l=1,l\neq k}^{N}\frac{1}{\lambda_k-\lambda_l}\begin{vmatrix} y_{1k} & y_{1l} \\ y_{2k} & y_{2l} \end{vmatrix}\begin{vmatrix} z_{1k} & z_{1l} \\ z_{2k} & z_{2l} \end{vmatrix}$$

Proposition 5.1:

i) $\{E_j^{(1)},E_k^{(1)}\}=0,\{E_j^{(1)},E_k^{(2)}\}=0,\{E_j^{(2)},E_k^{(2)}\}=0$

$$\forall j,k=1,2,\dots N \quad (26)$$

ii) $\{dE_j^{(l)},j=1,2,\dots N;l=1,2\}$ are the linear independence.

iii) $H_\mu=\sum_{j=1}^{N}\frac{1}{\mu-\lambda_j}(E_j^{(1)}+E_j^{(2)})=\sum_{m=0}^{\infty}\mu^{-m-1}h_m \quad (27)$

$$h_m=\sum_{j=1}^{N}\lambda_j^m(E_j^{(1)}+E_j^{(2)}) \quad m=0,1,2,\dots \quad (28)$$

Theorem 5.2: The Bargmann [8] systems (25) are the completely integrable systems in the Liouville sense. *i.e.*

$$\{h,E_j^{(l)}\}=0,\quad l=1,2;j=1,2,\dots,N \quad (29)$$

$$\{h_m,E_j^{(l)}\}=0,\quad l=1,2;j=1,2,\dots,N \quad (30)$$

$$\{h_m,h_n\}=0,\quad m,n=0,1,2,\dots \quad (31)$$

$$\{h,h_m\}=0,\quad m=0,1,2,\dots \quad (32)$$

Proof: By (26) and (28), we have

$$\{h_m,h_n\}=0,\quad m,n=0,1,2,\dots$$

from (27), then

$$\{H_\lambda,H_\mu\}=0$$

using $h=h_0$, we have

$$\{h,E_j^{(l)}\}=0,\quad l=1,2;j=1,2,\dots,N$$

□

According to the above Theorem, the Hamiltonian phase flows $g_{h_n}^{t_n}$ and $g_{h_m}^{t_m}$ are commutable.

Now, we arbitrarily choose an initial value

$$(y_i(0,0), z_i(0,0))^T \ i=1,2,$$

let

$$\begin{pmatrix} y_i(t_m,t_n) \\ z_i(t_m,t_n) \end{pmatrix} = g_{h_m}^{t_m} g_{h_n}^{t_n} \begin{pmatrix} y_i(0,0) \\ z_i(0,0) \end{pmatrix} = g_{h_n}^{t_n} g_{h_m}^{t_m} \begin{pmatrix} y_i(0,0) \\ z_i(0,0) \end{pmatrix}$$

From (8), (9) and Theorem 4.2, the following theorem holds.

Theorem 5.3: Suppose (y_1, y_2, z_1, z_2) is an involutive solution of the Hamiltonian [11] canonical equation systems (25), then

$$\begin{cases} q = <y_1, z_2> \\ p = <y_1, z_1> - \dfrac{1}{2} <y_1, z_2>^2 + \dfrac{1}{2} <\Lambda y_1, z_2> \end{cases}$$

satisfies the ($m+1$)-order long-wave equation (12).

Remark: By Theorem 5.2, soliton waves have the following properties: when two of them interact, the larger soliton has been shifted to the right of where it would have been no interaction, and the smaller shifted to the left by the same time (see Fig. 3) [22-24].
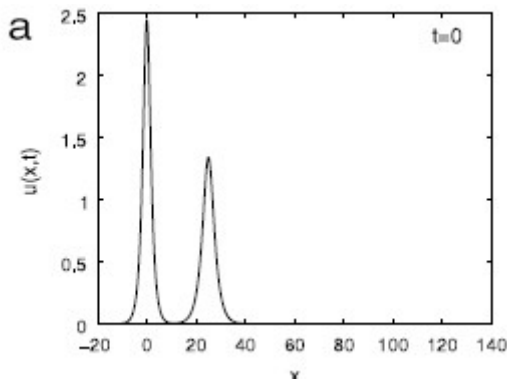
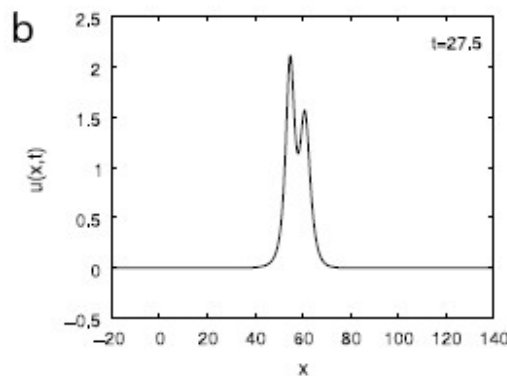Figure3. Interaction of two solitary waves at different times

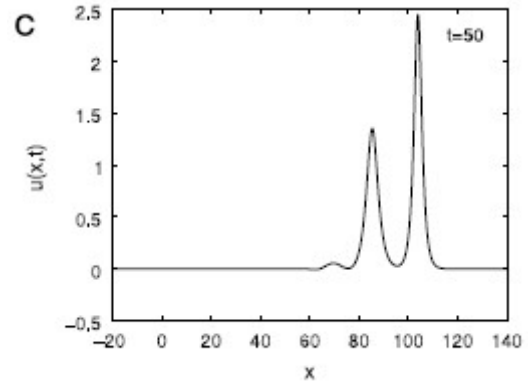Figure3. Interaction of two solitary waves at different times

Figure3. Interaction of two solitary waves at different times

Especially, if (y_1, y_2, z_1, z_2) satisfies

$$\begin{cases} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_x = \bar{M} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_x = -\bar{M}^T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t_1} = \bar{W}_1 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_{t_1} = -\bar{W}_1^T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{cases}$$

where

$$\bar{M} = \begin{pmatrix} \dfrac{1}{2}\Lambda - \dfrac{1}{2} <y_1, z_2> & E \\ \bar{M}_{21} & \dfrac{1}{2}\Lambda - \dfrac{1}{2} <y_1, z_2> \end{pmatrix}$$

$$\bar{M}_{21} = \dfrac{1}{4}\Lambda^2 + \dfrac{3}{4} <y_1, z_2>^2 - \dfrac{1}{2} <y_1, z_1> - \dfrac{1}{2} <y_2, z_2>$$

$$\bar{W}_1 = \begin{pmatrix} \bar{A}_1 & \bar{B}_1 \\ \bar{C}_1 & \bar{D}_1 \end{pmatrix}$$

$$\bar{A}_1 = \dfrac{1}{2}\Lambda^2 - \dfrac{1}{2} <\Lambda y_1, z_2> - <y_2, z_2>$$

$$\bar{B}_1 = \Lambda + <y_1, z_2>$$

$$\bar{C}_1 = <y_2, z_1> - \dfrac{1}{2}(<y_1, z_1> + <y_2, z_2>) \Lambda$$
$$- \dfrac{1}{4} <y_1, z_2> \Lambda^2 - \dfrac{1}{4} <\Lambda y_1, z_2> \Lambda + \dfrac{1}{4}\Lambda^3$$
$$- \dfrac{1}{4} <\Lambda^2 y_1, z_2> - \dfrac{1}{4} <\Lambda y_1, z_2> \Lambda$$
$$+ \dfrac{1}{2} <y_1, z_2> <\Lambda y_1, z_2> + \dfrac{1}{4} <y_1, z_2>^2 \Lambda$$

$$\bar{D}_1 = \dfrac{1}{2}\Lambda^2 - \dfrac{1}{2} <\Lambda y_1, z_2> - <y_1, z_1>$$

then

$$\begin{cases} q = <y_1, z_2> \\ p = <y_1, z_1> - \dfrac{1}{2} <y_1, z_2>^2 + \dfrac{1}{2} <\Lambda y_1, z_2> \end{cases}$$

satisfies long-wave equation

$$\begin{pmatrix} q_{t_1} \\ p_{t_1} \end{pmatrix} = \begin{pmatrix} 2p_x + q_{xx} + 2qq_x \\ -p_{xx} + 2(qp)_x \end{pmatrix}$$

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