Approximation of Common Fixed Points for Families of Mappings in Banach Space

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***Abstract*—In this paper. We introduce a general iterative method for the family of mappings and prove the strong convergence of the new iterative scheme in Banach space. The new iterative method includes the iterative scheme of Khan and Domlo and Fukhar-ud-din [Common fixed points Noor iteration for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. 341 (2008) 1–11]. The results generalize the corresponding results.**

***Index Terms*—strong convergence, common fixed point, generalized asymptotically quasi-nonexpansive mapping, viscosity iteration sequence, modified W-mapping**

I. INTRODUCTION

Let C be a nonempty subset of a real Banach space E and T a self-mapping of C. The set of fixed points of T denote by $F(T)$ and we assume that $F(T) \neq \phi$. The mapping T is said to be generalized asymptotically quasi-nonexpansive [1] if there exists two sequences $\{u_n\}$, $\{h_n\}$ in $[0,+\infty)$ with $\lim_{n\to\infty} u_n = 0$ and $\lim_{n\to\infty} h_n = 0$ such that

$$\left\|T^n x - p\right\| \le (1+u_n)\left\|x-p\right\| + h_n \ , \quad \forall\, x\in C\,, p\in F(T)\,, \tag{1.1}$$

where $n = 1,2,\ldots$. If $h_n = 0$ for all $n \ge 1$, then T becomes asymptotically quasi-nonexpansive mapping; if $u_n = 0$ and $h_n = 0$ for all $n \ge 1$, then T becomes quasi-nonexpansive mapping. It is known that an asymptotically nonexpansive mapping is an asymptotically quasi-nonexpansive.

Let C be a nonempty closed convex subset of a real Banach E , $\{S_n\}$ a family of generalized asymptotically nonexpansive mappings of C into itself and let $\{\alpha_{in} : n, i\in N, 1\le i\le k\}$ be a sequence of real numbers such that $0 \le a_{in} \le 1$ for every $n, i\in N$, $1\le i\le k$. Then we consider the following mapping of C into itself:

$$\begin{aligned} U_{1n} &= (1-\alpha_{1n})I + \alpha_{1n}S_1^n, \\ U_{2n} &= (1-\alpha_{2n})I + \alpha_{2n}S_2^n U_{1n}\,, \\ &\vdots \\ U_{(k-1)n} &= (1-\alpha_{(k-1)n})I + \alpha_{(k-1)n}S_{(k-1)}^n U_{(k-2)n}\,, \\ W_n = U_{kn} &= (1-\alpha_{kn})I + \alpha_{kn}S_k^n U_{(k-1)n}\,. \end{aligned} \tag{1.2}$$

Such a mapping W_n is called the modified W-mapping generated by $S_1, S_2, \ldots, S_k$ and $\alpha_{1n}, \alpha_{2n}, \ldots, \alpha_{kn}$ (See [4]).

In 2008, Khan, Domlo and Fukhar-ud-din [2] introduced the following iteration process for a family of asymptotically quasi-nonexpansive mappings, for an arbitrary $x_1 \in C$:

$$\begin{cases} y_{1n} = (1-\alpha_{1n})x_n + \alpha_{1n}S_1^n y_{0n}, \\ y_{2n} = (1-\alpha_{2n})x_n + \alpha_{2n}S_2^n y_{1n}\,, \\ \vdots \\ y_{(k-1)n} = (1-\alpha_{(k-1)n})x_n + \alpha_{(k-1)n}S_{k-1}^n y_{(k-2)n}\,, \\ x_{n+1} = (1-\alpha_{kn})x_n + \alpha_{kn}S_k^n y_{(k-1)n}, \end{cases} \tag{1.3}$$

where $y_{0n} = x_n$, $\alpha_{in} \subset [0,1]$ $(i = 1, 2, \ldots, k)$, $n = 1, 2, \ldots$ and proved that the iterative sequence $\{x_n\}$ defined by (1.3), converges strongly to a common fixed point of the family of mappings if and only if $\lim\inf_{n\to\infty} d(x_n, F) = 0$, where $d(x_n, F) = \inf_{p\in F}\left\|x-p\right\|$. (1.3) may denote by

$$x_{n+1} = W_n x_n\,, \tag{1.4}$$

where $\{S_i\}_{i=1}^k$ is a family of asymptotically quasi-nonexpansive mappings of C into itself.

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Recently, Chang, lee, chan and kim [3] introduced the following iteration process of asymptotically nonexpansive mappings in Banach space:

$$\begin{cases} x_n = \lambda_n f(x_n) + (1-\lambda_n) S^n y_n, \\ y_n = \beta_n x_n + (1-\beta_n) S^n x_n, \end{cases} \quad (1.5)$$

where $\{\lambda_n\}$, $\{\beta_n\} \subset [0,1]$, f is a fixed contractive mapping, and gave the sufficient and necessary condition for the iterative sequence $\{x_n\}$ converges to the fixed points of S.

For a family of mappings, it is quite significant to devise a general iteration scheme which extends the iteration (1.3) and the iteration (1.5), simultaneously. Thereby, to achieve this goal, we introduce a new iteration process for a family of mappings as follows:

Let C be a nonempty closed convex subset of a real Banach E, $\{S_i : C \to C, i = 1, 2, \ldots, k\}$ a family of generalized asymptotically quasi-nonexpansive mappings and $f : C \to C$ a fixed contractive mapping with contractive coefficient $\alpha \in [0,1]$. For a given $x_1 \in C$, the iteration scheme is defined as follows:

$$x_{n+1} = \lambda_n f(x_n) + (1-\lambda_n) W_n x_n, \quad (1.6)$$

where $\{\lambda_n\} \in [0,1]$. Further, let $\{T_n : C \to C, n \in N\}$ be a family of mappings. We propose the following iteration scheme:

$$x_{n+1} = \lambda_n f(x_n) + (1-\lambda_n) T_n x_n, \quad (1.7)$$

where $\{\lambda_n\} \in [0,1]$.

The purpose of this paper is to study the convergence problem of the iterative sequences $\{x_n\}$ defined by (1.6) and (1.7). The results extend the results of [2].

II. Preliminaries

Lemma 2.1 (see [5]) Let $\{a_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative real sequences satisfying the following conditions:

$$a_{n+1} = (1+\delta_n) a_n + \gamma_n, \ \forall\, n \in N,$$

where $\Sigma_{n=1}^{\infty} \delta_n < \infty$ and $\Sigma_{n=1}^{\infty} \gamma_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. Moreover, if in addition, $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Next, we introduce two new conditions. Let C be a nonempty closed convex subset of a real Banach E. Let T_n and $\mathcal{T}$ be families of mappings of C into itself such that $\phi \neq F(\mathcal{T}) \subset \bigcap_{n=1}^{\infty} F(T_n)$, where $F(T_n)$ is the set of all fixed points of T_n and $F(\mathcal{T})$ is the set of all common fixed points of $\mathcal{T}$. Then, T_n is said to satisfy:

(a) condition (I) with $\mathcal{T}$ if for $y \in F(\mathcal{T})$, there exists two sequences $\{\delta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ in $[0,\infty)$ with $\Sigma_{n=1}^{\infty} \delta_n < +\infty$, $\Sigma_{n=1}^{\infty} \gamma_n < +\infty$ such that

$$\|T_n x - y\| \le (1+\delta_n)\|x - y\| + \gamma_n, \ \forall\, x \in C, n \in N;$$

(b) condition (II) with $\mathcal{T}$ if condition (I) is satisfied and $F(\mathcal{T}) = \bigcap_{n=1}^{\infty} F(T_n)$.

Lemma 2.2 Let C be a nonempty closed convex subset of a real Banach E, T_n and $\mathcal{T}$ two families of mappings of C into itself such that $\phi \neq F(\mathcal{T}) \subset \bigcap_{n=1}^{\infty} F(T_n)$. Suppose that T_n satisfies condition (I) with $\mathcal{T}$. Let $\{\lambda_n\}$ be a sequence of real numbers with $0 \le \lambda_n \le 1$ for all $n \in N$, $\Sigma_{n=1}^{\infty} \lambda_n < \infty$. Let f : $C \to C$ be a contraction with $0 < \alpha < 1$. The sequence $\{x_n\}$ defined by (1.7), then

(1) there exists a sequence $\{\xi_n\}$ in $[0,\infty)$ with $\Sigma_{n=1}^{\infty} \xi_n < +\infty$ such that

$$\|x_{n+1} - p\| \le (1+\delta_n)\|x_n - p\| + \xi_n, \\ \forall\, p \in F(\mathcal{T}),\ \forall\, n \in N;$$

(2) there exists a constant $M_1 > 0$, such that

$$\|x_{n+m} - p\| \le M_1 \|x_n - p\| + M_1 \sum_{j=n}^{\infty} \xi_j, \\ \forall\, p \in F(\mathcal{T}),\ \forall\, n, m \in N.$$

Proof. (1) Let $p \in F(\mathcal{T})$, by (1.7), we have

$$\begin{aligned} &\|x_{n+1} - p\| \\ &\le \lambda_n \|f(x_n) - p\| + (1-\lambda_n)\|T_n x_n - p\| \\ &\le \lambda_n \alpha \|x_n - p\| + \lambda_n \|f(p) - p\| \\ &\quad + (1-\lambda_n)\{(1+\delta_n)\|x_n - p\| + \gamma_n\} \\ &\quad = \{\lambda_n \alpha + (1-\lambda_n)(1+\delta_n)\}\|x_n - p\| \\ &\qquad + (1-\lambda_n)\gamma_n + \lambda_n \|f(p) - p\| \\ &\quad \le \{\lambda_n \alpha + (1-\lambda_n)(1+\delta_n)\}\|x_n - p\| \\ &\qquad + \gamma_n + \lambda_n \|f(p) - p\| \\ &\le (1+\delta_n)\|x_n - p\| + \xi_n, \end{aligned} \quad (2.1)$$

where $\xi_n = \gamma_n + \lambda_n \|f(p) - p\|$, $\Sigma_{n=1}^{\infty} \xi_n < +\infty$. This completes the proof of (1).

(2) If $t \geq 0$, then $1+t \leq e^t$. And for all integer $m \geq 1$, by (2.1), we obtain

$$\begin{aligned}
\|x_{n+m}-p\| &\leq (1+\delta_{n+m-1})\|x_{n+m-1}-p\|+\xi_{n+m-1} \\
&\leq (1+\delta_{n+m-1})\{(1+\delta_{n+m-2})\|x_{n+m-2}-p\| \\
&\quad +\xi_{n+m-2}\}+\xi_{n+m-1} \\
&\leq \prod_{j=n+m-2}^{n+m-1}(1+\delta_j)\|x_{n+m-2}-p\| \\
&\quad +(1+\delta_{n+m-1})(\xi_{n+m-2}+\xi_{n+m-1}) \\
&\leq \prod_{j=n+m-2}^{n+m-1}(1+\delta_j)[(1+\delta_{n+m-3})\|x_{n+m-3}-p\| \\
&\quad +\xi_{n+m-3}]+(1+\delta_{n+m-1})(\xi_{n+m-2}+\xi_{n+m-1}) \\
&\leq \prod_{j=n+m-3}^{n+m-1}(1+\delta_j)\|x_{n+m-3}-p\| \\
&\quad +\prod_{j=n+m-2}^{n+m-1}(1+\delta_j)\sum_{j=n+m-3}^{n+m-1}\xi_j \\
&\ \ \vdots \\
&\leq \prod_{j=n}^{n+m-1}(1+\delta_j)\|x_n-p\| \\
&\quad +\prod_{j=n+1}^{n+m-1}(1+\delta_j)\sum_{j=n}^{n+m-1}\xi_j \\
&\leq \exp\left(\sum_{j=n}^{n+m-1}\delta_j\right)\|x_n-p\| \\
&\quad +\exp\left(\sum_{j=n+1}^{n+m-1}\delta_j\right)\sum_{j=n}^{n+m-1}\xi_j \\
&\leq M_1\|x_n-p\|+M_1\sum_{j=n}^{\infty}\xi_j,
\end{aligned}$$

where $M_1=\exp(\Sigma_{j=1}^{\infty}\delta_j)$. This completes the proof.

Lemma 2.3 Let C be a nonempty closed convex subset of a real Banach E. Let S_i $(i=1,2,\ldots,k)$ be k generalized asymptotically quasi-nonexpansive self-mappings of C with $u_{in}, h_{in} \subset [0,\infty)$ such that $\Sigma_{n=1}^{\infty}u_{in}<+\infty$ and $\Sigma_{n=1}^{\infty}h_{in}<+\infty$ for all $i\in N$, $1\leq i\leq k$. Suppose $\bigcap_{i=1}^{k}F(S_i)\neq\phi$ and $\{\alpha_{in}\}\subset[0,1]$ $(i=1,2,\ldots,k)$ for all $n\in N$. Let W_n be the modified W-mapping generated by $S_1,S_2,\ldots,S_k$ and $\alpha_{1n},\alpha_{2n},\ldots,\alpha_{kn}$ for all $n\in N$. Then W_n satisfies condition (I) with $\{S_i\}_{i=1}^{k}$.

Proof. From (1.2) we obtain that $\bigcap_{i=1}^{k}F(S_i)\subset F(W_n)$. Let $\nu_n=\max_{1\leq i\leq k}u_{in}$, for all $n\in N$. Since $\Sigma_{n=1}^{\infty}u_{in}<+\infty$ for each i, therefore $\Sigma_{n=1}^{\infty}\nu_n<+\infty$. For all $p\in\bigcap_{i=1}^{k}F(S_i)$ and $x\in C$, it follows from (1.2) that

$$\begin{aligned}
\|U_{1n}x-p\| &\leq (1-\alpha_{1n})\|x-p\|+\alpha_{1n}\|S_1^n x-p\| \\
&\leq (1-\alpha_{1n})\|x-p\|+\alpha_{1n}\{(1+u_{1n})\|x-p\|+h_{1n}\} \\
&\leq (1+u_{1n})\|x-p\|+h_{1n} \\
&\leq (1+\nu_n)\|x-p\|+h_{1n}.
\end{aligned}$$

Assume that

$$\|U_{jn}x-p\|\leq(1+\nu_n)^j\|x-p\|+(1+\nu_n)^{j-1}\Sigma_{i=1}^{j}h_{in},$$

holds for some $1\leq j\leq k-1$. Then

$$\begin{aligned}
\|U_{(j+1)n}x-p\| &\leq (1-\alpha_{(j+1)n})\|x-p\| \\
&\quad +\alpha_{(j+1)n}\|S_{j+1}^n U_{jn}x-p\| \\
&\leq (1-\alpha_{(j+1)n})\|x-p\|+\alpha_{(j+1)n} \\
&\quad \times\{(1+u_{(j+1)n})\|U_{jn}x-p\|+h_{(j+1)n}\} \\
&\leq (1-\alpha_{(j+1)n})\|x-p\|+\alpha_{(j+1)n} \\
&\quad \times(1+u_{(j+1)n})\{(1+\nu_n)^j\|x-p\| \\
&\quad +(1+\nu_n)^{j-1}\sum_{i=1}^{j}h_{in}\}+\alpha_{(j+1)n}h_{(j+1)n} \\
&\leq \{(1-\alpha_{(j+1)n})+\alpha_{(j+1)n}(1+\nu_n)^{j+1}\} \\
&\quad \times\|x-p\|+(1+\nu_n)^j\sum_{i=1}^{j}h_{in}+h_{(j+1)n} \\
&\leq (1+\nu_n)^{j+1}\|x-p\|+(1+\nu_n)^j\sum_{i=1}^{j+1}h_{in}
\end{aligned}$$

Thus, by induction, we have

$$\begin{aligned}
\|U_{jn}x-p\| &\leq (1+\nu_n)^j\|x-p\|+(1+\nu_n)^{j-1}\sum_{i=1}^{j}h_{in} \\
&\leq \left\{1+\sum_{r=1}^{j}\frac{j!}{r!(j-r)!}\nu_n^r\right\}\|x-p\| \\
&\quad +e^{\nu_n(j-1)}\sum_{i=1}^{j}h_{in} \\
&=(1+\delta_{jn})\|x-p\|+\gamma_{jn}, \\
&\qquad j=1,2,\ldots,k,
\end{aligned} \tag{2.2}$$

where, $\delta_{jn}=\sum_{r=1}^{j}\frac{j!}{r!(j-r)!}\nu_n^r$, $\gamma_{jn}=M_j\sum_{i=1}^{j}h_{in}$, $M_j=\sup_n e^{\nu_n(j-1)}$. Since $\Sigma_{n=1}^{\infty}\nu_n<+\infty$ and $\Sigma_{n=1}^{\infty}h_{in}<+\infty$ for all $1\le i\le k$, therefore $\Sigma_{n=1}^{\infty}\delta_{jn}<+\infty$ and $\Sigma_{n=1}^{\infty}\gamma_{jn}<+\infty$ $(1\le j\le k)$. Hence,

$$\|W_n x-p\|=\|U_{kn}x-p\| \le(1+\delta_{kn})\|x-p\|+\gamma_{kn}.$$

Thus, W_n satisfies condition (I) with $\{S_i\}_{i=1}^{k}$.

Lemma 2.1 in [2] can be easily elicited from lemma 2.2 and lemma 2.3.

III. Strong Convergence Theorems for the Family of Mappings

Theorem 3.1 If all the condition of lemma 2.2 are conformed, and plus one another that $F(\mathcal{T})$ be a closed subset of E. Then the sequence $\{x_n\}$ defined by (1.7) converges strongly to $p\in F(\mathcal{T})\subset\bigcap_{n=1}^{\infty}F(T_n)$ if and only if $\liminf_{n\to\infty}d(x_n,F(\mathcal{T}))=0$, where $d(x_n,F(\mathcal{T}))=\inf_{y\in F(\mathcal{T})}\|x_n-y\|$.

Proof: We will only prove the sufficiency; the necessity is obvious. From Lemma2.2 (1), we have

$$d(x_{n+1},F(\mathcal{T}))\le(1+\delta_n)d(x_n,F(\mathcal{T}))+\xi_n,\quad \forall\ p\in F(\mathcal{T}),\quad n>1.$$

By Lemma 2.1 and $\liminf_{n\to\infty}d(x_n,F(\mathcal{T}))=0$, we get that $\lim_{n\to\infty}d(x_n,F(\mathcal{T}))=0$. Next, we prove that $\{x_n\}$ is a Cauchy sequence. From Lemma 2.2 (2), we have

$$\|x_{n+m}-p\|\le M_1\|x_n-p\|+M_1\sum_{j=n}^{\infty}\xi_j ,\quad \forall\ n,m\in N,\quad p\in F(\mathcal{T})$$

Hence, for all integer $m\ge1$ and all $p\in F(\mathcal{T})$,

$$\begin{aligned}\|x_{n+m}-x_n\|&\le\|x_{n+m}-p\|+\|x_n-p\|\\&\le(M_1+1)\|x_n-p\|+M_1\sum_{j=n}^{\infty}\xi_j .\end{aligned}\tag{3.1}$$

Taking infimum over $p\in F(\mathcal{T})$ in (3.1) gives

$$\begin{aligned}\|x_{n+m}-x_n\|&\le(M_1+1)d(x_n,F(\mathcal{T}))\\&\quad+M_1\sum_{j=n}^{\infty}\xi_j .\end{aligned}\tag{3.2}$$

Now, since $\lim_{n\to\infty}d(x_n,F(\mathcal{T}))=0$ and $\Sigma_{j=1}^{\infty}\xi_j<+\infty$, given $\varepsilon>0$, there exists an integer $N_1>0$ such that for all $n>N_1$, $d(x_n,F(\mathcal{T}))<\frac{\varepsilon}{2(M_1+2)}$ and $\sum_{j=n}^{\infty}\xi_n<\frac{\varepsilon}{2(M_1+2)}$. So for all integers $n>N_1$, $m\ge1$, we obtain from (3.2) that

$$\|x_{n+m}-x_n\|\le\varepsilon,\quad n>N_1,m\ge1.$$

Hence, $\{x_n\}$ is a Cauchy sequence in E. Since E is complete, there exists $q\in E$ such that $\lim_{n\to\infty}x_n=q$ .We now show that $q\in F(\mathcal{T})$. Since $d(x_n,F(\mathcal{T}))\to0$ and $x_n\to q$, as $n\to\infty$, for each $\epsilon>0$, there exists a natural number N such that for all $n>N$, $d(x_n,F(\mathcal{T}))<\frac{\epsilon}{2}$ and $\|x_n-q\|<\frac{\epsilon}{2}$. In particular, we have $d(x_N,F(\mathcal{T}))=\inf_{p\in F(\mathcal{T})}\|x_N-p\|<\frac{\epsilon}{2}$, i.e., there exists $p\in F(\mathcal{T})$, such that $\|x_N-p\|<\frac{\epsilon}{2}$, hence

$$\|q-p\|\le\|x_N-q\|+\|x_N-p\|<\epsilon .$$

Since $F(\mathcal{T})$ is a closed subset of E , we obtain $q\in F(\mathcal{T})$. This completes the proof.

Using Theorem 3.1, we obtain the following theorem:

Theorem 3.2 Let C be a nonempty closed convex subset of a real Banach E , T_n and $\mathcal{T}$ two families of mappings of C into itself and $\bigcap_{n=1}^{\infty}F(T_n)$ a closed subset of E. Suppose that T_n satisfies condition (II) with $\mathcal{T}$. Let $\{\lambda_n\}$ be a sequence of real numbers with $0\le\lambda_n\le1$ for all $n\in N$, and $f:\ C\to C$ a contraction with $0<\alpha<1$. Suppose that $x_1\in C$, $\Sigma_{n=1}^{\infty}\lambda_n<\infty$. Then the sequence $\{x_n\}$ defined by (1.7) converges strongly to $p\in\bigcap_{n=1}^{\infty}F(T_n)$ if and only if $\liminf_{n\to\infty}d(x_n,\bigcap_{n=1}^{\infty}F(T_n))=0$.

IV. Strong Convergence Theorems for Generalized Asymptotically Quasi-Nonexpansive Mapping

Theorem 4.1 If all the condition of lemma 2.3 are conformed. Let $\bigcap_{n=1}^{\infty}F(S_n)$ be a closed subset of E. Let $\{\lambda_n\}$ be a sequence of real numbers with

$0 \le \lambda_n \le 1$ for all $n \in N$, $\Sigma_{n=1}^{\infty}\lambda_n < \infty$. Let $f: C \to C$ be a contraction with $0 < \alpha < 1$. Starting from arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ defined by (1.6), then the sequence $\{x_n\}$ converges strongly to $p \in \bigcap_{i=1}^{k} F(S_i)$ if and only if $\liminf_{n\to\infty} d(x_n, \bigcap_{i=1}^{k} F(S_i)) = 0$.

Proof: By Lemma 2.3, W_n satisfies condition (I) with $\{S_i\}_{i=1}^{k}$, we obtain from Theorem 3.1 that $\{x_n\}$ converges strongly to $p \in \bigcap_{i=1}^{k} F(S_i)$. This completes the proof.

Using Theorem 4.1, we also obtain the following theorem which was proved by Khan, Domlo, and Fukhar-ud-din [2].

Theorem 4.2 Let C be a nonempty closed convex subset of a real Banach E. Let S_i $(i = 1, 2, \ldots, k)$ be k asymptotically quasi-nonexpansive self-mappings of C with $u_{in} \subset [0, \infty)$ such that $\Sigma_{n=1}^{\infty} u_{in} < +\infty$ for all $i \in N$, $1 \le i \le k$. Suppose $\bigcap_{i=1}^{k} F(S_i) \ne \phi$ and $\{\alpha_{in}\} \subset [0,1]$ $(i = 1, 2, \ldots, k)$ for all $n \in N$. For any given $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (1.3). Then $\{x_n\}$ converges strongly to $p \in \bigcap_{i=1}^{k} F(S_i)$ if and only if $\liminf_{n\to\infty} d(x_n, \bigcap_{i=1}^{k} F(S_i)) = 0$.

Proof: Easy to show that $\bigcap_{i=1}^{k} F(S_i)$ is closed. In (1.6), taking , $\lambda_n = 0$ and $h_{in} = 0$ $(i = 1,2,\ldots,k)$ for all $n \in N$, (1.6) is reduced to (1.3). Therefore the conclusion of Theorem 4.2 can be obtained from Theorem 4.1 immediately.

Theorem 4.3 Let C be a nonempty closed convex subset of a real Banach E. Let S be a asymptotically nonexpansive self-mappings of C with $u_n \subset [0, \infty)$, i.e., $\|S^n x - S^n y\| \le (1 + u_n)\|x - y\|$ for all $x, y \in C$, where $\Sigma_{n=1}^{\infty} u_n < +\infty$, suppose $F(S) \ne \phi$. Let $\{\lambda_n\}$ be a sequence of real numbers with $0 \le \lambda_n \le 1$ such that $\Sigma_{n=1}^{\infty}\lambda_n < \infty$ for all $n \in N$, and $f: C \to C$ a contraction with $0 < \alpha < 1$. Starting from arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ defined by (1.5). Then the sequence $\{x_n\}$ converges strongly to $p \in F(S)$ if and only if $\liminf_{n\to\infty} d(x_n, F(S)) = 0$.

Proof: In (1.6), taking $S_1 = S_2, = \cdots = S_{k-2} = I$, $\alpha_{kn} = 1$, $1 - \alpha_{(k-1)n} = \beta_n$, $S_k = S_{K-1} = S$, $h_{in} = 0$ $(i = 1, 2, \ldots, k)$ for all $n \in N$, (1.6) is reduced to (1.5). Therefore the conclusion of Theorem 4.3 can be obtained from Theorem 4.1 immediately.

Let $G = \{g : [0, \infty) \to [0, \infty) : g(0) = 0,$ g: continuous; strictly increasing; convex$\}$. We have the following lemma for a uniformly convex Banach space.

Lemma 4.1 (see [6]) E is a uniformly convex Banach space if and only if for every bounded subset B of E, there exists $g \in G$ such that

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x - y\|)$$

For all $x, y \in B$ and $\lambda \in [0, 1]$.

Lemma 4.2 Let E be a real uniformly convex Banach space and C a nonempty bounded closed convex subset of E. Let $S_i : C \to C$ $(i = 1, 2, \ldots, k)$ be k uniformly L-Lipschitzian, generalized asymptotically quasi-nonexpansive mappings with $\{u_{in}\}, \{h_{in}\} \subset [0, \infty)$ such that $\Sigma_{n=1}^{\infty} u_{in} < +\infty$ and $\Sigma_{n=1}^{\infty} h_{in} < +\infty$ for all $i \in N$, $1 \le i \le k$. Let $\{\alpha_{in}\}$ ($i, n \in N$, $1 \le i \le k$) be a sequence of real numbers with $0 < a \le \alpha_{kn} \le 1$, $0 < b \le \alpha_{in} \le c < 1$ ($1 \le i \le k-1$) for all $n \in N$ and W_n be a modified W-mapping generated by $\{S_i\}_{i=1}^{k}$ and $\{\alpha_{in}\}_{i=1}^{k}$. Suppose $\bigcap_{i=1}^{k} F(S_i) \ne \phi$, then $\bigcap_{i=1}^{k} F(S_i) = \bigcap_{n=1}^{\infty} F(W_n)$.

Proof: $\bigcap_{i=1}^{k} F(S_i) \subset \bigcap_{n=1}^{\infty} F(W_n)$ is obvious. Let $p \in \bigcap_{n=1}^{\infty} F(W_n)$, we obtain from (1.2) that $W_n p = (1 - \alpha_{kn})p + \alpha_{kn} S_k^n U_{(k-1)n} p$. Since $0 < a \le \alpha_{kn} \le 1$ for all $n \in N$, therefore $p = S_k^n U_{(k-1)n} p$. Let $z \in \bigcap_{n=1}^{k} F(S_n)$, we have

$$\begin{aligned} \|p - z\|^2 &\le \left\{\|p - S_i^n U_{(i-1)n} p\| + \|S_i^n U_{(i-1)n} p - z\|\right\}^2 \\ &= \|p - S_i^n U_{(i-1)n} p\| \left\{\|p - S_i^n U_{(i-1)n} p\| \right. \\ &\quad \left. + 2\|S_i^n U_{(i-1)n} p - z\|\right\} + \|S_i^n U_{(i-1)n} p - z\|^2 \\ &\le M_2 \|p - S_i^n U_{(i-1)n} p\| \\ &\quad + \left\{(1 + u_{in})\|U_{(i-1)n} p - z\| + h_{in}\right\}^2 \end{aligned}$$

$$= (1+u_{in})^2 \|U_{(i-1)n}p - z\|^2 + 2(1+u_{in})h_{in}\|U_{(i-1)n}p - z\| + M_2\|p - S_i^n U_{(i-1)n}p\| + h_{in}^2$$
$$\le (1+u_{in})^2 \|(1-\alpha_{(i-1)n})(p-z) + \alpha_{(i-1)n}\{S_{i-1}^n U_{(i-2)n}p - z\}\|^2 + M_3 h_{in} + M_2\|p - S_i^n U_{(i-1)n}p\| + h_{in}^2, \quad (4.1)$$

where,

$$M_2 = \sup_{n\in N}\{\|p - S_i^n U_{(i-1)n}p\| + 2\|S_i^n U_{(i-1)n}p - z\|\}$$
$$M_3 = \sup_{n\in N}\{2(1+u_{in})\|U_{(i-1)n}p - z\|\}.$$

By Lemma 4.1 and (4.1), we get

$$\|p-z\|^2 \le (1+u_{in})^2\{(1-\alpha_{(i-1)n})\|p-z\|^2 + \alpha_{(i-1)n}\|S_{i-1}^n U_{(i-2)n}p - z\|^2 - (1-\alpha_{(i-1)n})\alpha_{(i-1)n} \times g(\|S_{i-1}^n U_{(i-2)n}p - p\|)\} + M_2\|p - S_i^n U_{(i-1)n}p\| + M_3 h_{in} + h_{in}^2$$
$$\le (1+u_{in})^2\{(1-\alpha_{(i-1)n})\|p-z\|^2 + \alpha_{(i-1)n}[(1+u_{(i-1)n})\|U_{(i-2)n}p - z\| + h_{(i-1)n}]^2\} - (1+u_{in})^2(1-\alpha_{(i-1)n}) \times \alpha_{(i-1)n} g(\|S_{i-1}^n U_{(i-2)n}p - p\|) + M_2\|p - S_i^n U_{(i-1)n}p\| + M_3 h_{in} + h_{in}^2. \quad (4.2)$$

By (2.2) and (4.2), we have

$$\|p-z\|^2 \le (1+u_{in})^2(1-\alpha_{(i-1)n})\|p-z\|^2 + (1+u_{in})^2\alpha_{(i-1)n}\{(1+u_{(i-1)n}) \times [(1+\delta_{(i-2)n}\|p-z\| + \gamma_{(i-2)n}] + h_{(i-1)n}\}^2 - (1+u_{in})^2(1-\alpha_{(i-1)n}) \times \alpha_{(i-1)n} g(\|S_{i-1}^n U_{(i-2)n}p - p\|) + M_2\|p - S_i^n U_{(i-1)n}p\| + M_3 h_{in} + h_{in}^2. \quad (4.3)$$

If $i = k$, there is $\lim_{n\to\infty}\|p - S_i^n U_{(i-1)n}p\| = \lim_{n\to\infty} u_{in} = \lim_{n\to\infty}\delta_{(i-2)n} = \lim_{n\to\infty}\gamma_{(i-2)n} = \lim_{n\to\infty} h_{in} = \lim_{n\to\infty} u_{(i-1)n} = 0$, $0 < b \le \alpha_{(i-1)n} \le c < 1$, taking limit in (4.3), we have

$$\lim_{n\to\infty} g(\|S_{k-1}^n U_{(k-2)n}p - p\|) = 0,$$

hence, $\lim_{n\to\infty}\|S_{k-1}^n U_{(k-2)n}p - p\| = 0$. By the same token, we have

$$\lim_{n\to\infty}\|S_m^n U_{(m-1)n}p - p\| = 0, \quad m = 2, \dots, k.$$

Since each S_i is uniformly L-Lipschitzian, we conclude that

$$\|p - S_m^n p\| \le \|p - S_m^n U_{(m-1)n}p\| + \|S_m^n U_{(m-1)n}p - S_m^n p\|$$
$$\le \|p - S_m^n U_{(m-1)n}p\| + L\|U_{(m-1)n}p - p\|$$
$$= \|p - S_m^n U_{(m-1)n}p\| + L\alpha_{(m-1)n}\|p - S_{m-1}^n U_{(m-2)n}p\|,$$

hence, we get $\lim_{n\to\infty}\|p - S_m^n p\| = 0$ for all $m = 2, \dots, k$. Repeat the above process, we get $\lim_{n\to\infty}\|p - S_1^n p\| = 0$. Since

$$\|p - S_m p\| \le \|S_m p - S_m^{n+1}p\| + \|S_m^{n+1}p - p\|$$
$$\le L\|p - S_m^n p\| + \|S_m^{n+1}p - p\|$$
$$\to 0, \quad n \to \infty.$$

So, we obtain $\|p - S_m p\| = 0$ for $m = 1, 2, \dots, k$, i.e., $p = S_m p$, $\bigcap_{i=1}^k F(S_i) = \bigcap_{n=1}^\infty F(W_n)$. This completes the proof.

Theorem 4.4 Let E be a real uniformly convex Banach space, C a nonempty bounded closed convex subset of E and $S_i: C \to C$ $(i = 1, 2, \dots, k)$ be k uniformly L-Lipschitzian, generalized asymptotically quasi-nonexpansive mappings with $\{u_{in}\}, \{h_{in}\} \subset [0, \infty)$ such that $\Sigma_{n=1}^\infty u_{in} < +\infty$ and $\Sigma_{n=1}^\infty h_{in} < +\infty$ for all $i \in N$, $1 \le i \le k$. Suppose $\bigcap_{i=1}^k F(S_i) \ne \phi$ is closed. Let $\{\alpha_{in}\}$ ($i, n \in N$, $1 \le i \le k$) be a sequence of real numbers with $0 < a \le \alpha_{kn} \le 1$, $0 < b \le \alpha_{in} \le c < 1$ ($1 \le i \le k-1$) for all $n \in N$ and W_n be a modified W-mapping generated by $\{S_i\}_{i=1}^k$ and $\{\alpha_{in}\}_{i=1}^k$, $\{\lambda_n\}$ a sequence of real numbers with $0 \le \lambda_n \le 1$ for all $n \in N$, $\Sigma_{n=1}^\infty \lambda_n < +\infty$ and $f: C \to C$ a contraction with $0 < \alpha < 1$. For any given $x_1 \in C$, the sequence $\{x_n\}$ defined by the recursion (1.6). Then $\{x_n\}$

converges strongly to $p \in \bigcap_{n=1}^{\infty} F(W_n)$ if and only if $\liminf_{n\to\infty} d(x_n, \bigcap_{i=1}^{k} F(S_i)) = 0$.

Proof: By Lemma 4.2, there exists $\{S_i\}_{i=1}^{k}$ such that $\bigcap_{i=1}^{k} F(S_i) = \bigcap_{n=1}^{\infty} F(W_n)$, using Theorem 4.1, we obtain Theorem 4.4.

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