Highly Complex Chaotic System with Piecewise Linear Nonlinearity and Compound Structures

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Abstract—A new chaotic system is presented using a single parameter for a two-scroll attractor with high complexity, high chaoticity and widely chaotic range. The system employs two quadratic nonlinearities and two piecewise-linear nonlinearities. The high chaoticity is measured by the maximum Lyapunov Exponent of 0.429 and the high complexity is measured by the Kaplan-Yorke dimension of 2.3004. Dynamic properties are described in terms of symmetry, a dissipative system, an existence of attractor, equilibria, Jacobian matrices, bifurcations, Poincaré maps, chaotic waveforms, chaotic spectrum, and forming mechanisms of compound structures.

Index Terms—chaos, high-complexity, high-chaoticity, two-scroll attractor, piecewise-linear, compound structure

I. INTRODUCTION

The discovery of the eminent Lorenz system [1] has led to an extensive study of chaotic behaviors in nonlinear systems due to many possible applications in science and technology. The Lorenz system has seven terms in three-dimensional quadratic autonomous Ordinary Differential Equations (ODEs). Considerable research interests have also been made in exploring new chaotic systems either with simpler algebraic structures reflected by fewer terms in ODEs or with more chaotic and complex dynamical behaviors. A measure of chaoticity has been based on the Kaplan-Yorke dimension (D_{KY}), whilst a measure of complexity (or strangeness) has been based on the maximum positive Lyapunov Exponent (LE).

Several chaotic systems have been suggested to retain two quadratic nonlinearities with fewer terms in ODEs and the attractors closely resemble the two-scroll Lorenz attractor. Examples of systems with six terms in ODEs include, Lu and Chen [2], Zhou et al. [3], Li et al. [4], Lui and Yang [5], Pehlivan and Glu [6], and Pan et al. [7] systems. Examples of systems with five terms in ODEs include Sprott cases B and C [8], the diffusionless Lorenz system [9], and the five-term system [10]. Other systems have alternatively employed simpler piecewise-linear nonlinearities with five-terms in ODEs found in the jerk model [11], or a piecewise-linear Lorenz system [12]. Both systems are, however, lack of rotation symmetry [13]. Among all these systems, Sun and Sprott [14] have reported that the diffusionless Lorenz system has potentially possessed the highest value of D_{KY} at 2.2354 through a single control system parameter. It can be considered that this diffusionless Lorenz system is simple in terms of algebraic structure with very high complexity in terms of D_{KY} value.

Recently, two-scroll or many-scroll chaotic systems have employed more than two quadratic nonlinearities for generating complex chaotic attractors. For example, Li and Ou system [15] have employed three quadratic nonlinearities for a two-scroll complex attractor. Wang system [16] has employed three quadratic nonlinearities with eight-terms in ODEs for three-scroll and four-scroll attractors. Dadras et al. system [17] has realized four quadratic nonlinearities with eight-terms in ODEs for two-scroll, three-scroll and four-scroll attractors. Although many-scroll attractors have successfully been generated at the expense of extra three or four quadratic nonlinearities and a number of control parameters, the values of LE and D_{KY} are only somewhat comparable to those of chaotic systems with two quadratic nonlinearities. In particular, the requirements for extra quadratic nonlinearities are not well suited to circuit implementations.

In this paper, a new chaotic system is presented using a single parameter for a two-scroll attractor with high complexity, high chaoticity and widely chaotic range. The system employs two quadratic nonlinearities and two piecewise-linear nonlinearities, including tanh and absolute value nonlinearity. Dynamic properties are theoretically and numerically described in terms of symmetry, a dissipative system, an existence of attractor, equilibria, Jacobian matrix, bifurcation diagram with period-doubling route to chaos, Poincaré maps, chaotic waveforms in time domain, frequency spectrum, and forming mechanisms of compound structures.

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II. PROPOSED HIGHLY COMPLEX CHAOTIC SYSTEM WITH PIECEWISE-LINEAR NONLINEARITY

A new chaotic system is expressed as a set of three first-order autonomous differential equations with six-terms in ODEs as follows:

\[
\begin{aligned}
\dot{x} &= y - x \\
\dot{y} &= -z \tanh(x) \\
\dot{z} &= -R + xy + |y|
\end{aligned}
\]  

(1)

where \(x, y, z\) are system state variables and \(R\) is a single adjustable positive parameter. This new system (1) has been constructed through the realization of two quadratic and two piecewise-linear nonlinearities. The \(\tanh\) and absolute value functions were found as potential nonlinearities that sustain two-scroll attractor topology similar to Lorenz attractor, but foster higher values of the positive parameter \(R\) in the case of the equilibrium point \(P^\prime\) is equal to

\[
\begin{aligned}
&\dot{x} = y - x \\
&\dot{y} = -z \tanh(x) \\
&\dot{z} = -R + xy + |y|
\end{aligned}
\]  

(2)

As \(p<0\), the system (1) is dissipative system with an exponential rate of contraction as

\[
\frac{dV}{dt} = \exp(p) = \exp(-1).
\]  

(3)

In other words, a volume element \(V_0\) becomes smaller by the flow into a volume element \(\exp(-t)\) in time \(t\). Each volume containing the system trajectories shrinks to zero as \(t\to\infty\) at an exponential rate of \(-1\). System orbits are ultimately confined into a specific limit set of zero volume, and the asymptotic motion settles onto an attractor. In conclusion, the existence of attractor is constant and independent to all of nonlinear terms.

B. Equilibria

The equilibria of the system (1) can be found by using

\[
\begin{aligned}
y - x &= 0 \\
-z \tanh(x) &= 0 \\
-R + xy + |y| &= 0
\end{aligned}
\]  

(4)

Consequently, the system possesses two equilibrium points, i.e.

\[
\begin{aligned}
P^+(x_E, y_E, z_E) &= \left( \frac{-1}{2} + \frac{\sqrt{1 + 4R}}{2}, \frac{-1}{2} \sqrt{1 + 4R}, 0 \right), \\
P^-(x_E, y_E, z_E) &= \left( \frac{-1}{2} - \frac{\sqrt{1 + 4R}}{2}, \frac{-1}{2} \sqrt{1 + 4R}, 0 \right).
\end{aligned}
\]  

(5)

It is seen from (5) that the single parameter \(R\) determines the existence of equilibria. The two equilibrium points are almost symmetric, especially when \(R\) is large, and the two-scroll attractor topology can be obtained. However, the equilibrium point at the origin as found in the Lorenz system does not exist in (5) and therefore the system (1) rather has a strong uniform flow in \(+z\) direction.

C. Jacobian Matrices and Stability

The Jacobian matrix \((J)\) of partial derivatives of the system (1) is defined as

\[
J = \begin{bmatrix}
-1 & 1 & 0 \\
-z \tanh'(x) - z & 0 & -\tanh(x) \\
0 & x + \text{sign}(x) & 0
\end{bmatrix}.
\]  

(6)

Applying the two equilibrium points described in (5) into this Jacobian matrix one at a time and analyzing \(|J-J|=0\) reveal a similar result of a characteristic polynomial, i.e.

\[
J = \lambda^3 + \lambda^2 + R'\lambda + 2R' = 0,
\]  

(7)

where \(R'\) in the case of the equilibrium point \(P^+\) is equal to

\[
R' = R_+ = \left( \frac{1}{2} - \frac{\sqrt{1 + 4R}}{2} \right) \tanh \left( \frac{-1}{2} \sqrt{1 + 4R} \right)
\]  

(8)

and \(R'\) in the case of the equilibrium point \(P^-\) is equal to

\[
R' = R_- = \left( \frac{1}{2} - \frac{\sqrt{1 + 4R}}{2} \right) \tanh \left( \frac{-1}{2} \sqrt{1 + 4R} \right).
\]  

(9)

Applying the Routh-Hurwitz criterion to (7) yields

\[
R' - 2R' = -R' < 0.
\]  

(10)

It is verified from (10) that the parameter \(R\) must be positive for both equilibrium points in (5) so that the equilibrium points \(P^+\) and \(P^-\) satisfy the unstable condition of Routh-Hurwitz criterion. In addition, the form of eigenvalues of the characteristic polynomial in (7) can only be one negative real eigenvalue \(\lambda_1\) and a pair of complex conjugate eigenvalues \(\lambda_2\) and \(\lambda_3\) with positive real parts. Therefore, the above analyses show that the two equilibrium points \(P^+\) and \(P^-\) are all unstable, and can be classified as saddle focus nodes.

IV. NUMERICAL ANALYSIS

A. Bifurcations, Lyapunov Exponents, and Kaplan-Yorke Dimension

Numerical simulations have been performed in MATLAB using the initial condition of \((x_0, y_0, z_0) = (1, 1, 1)\). In fact, the initial condition is not crucial, and can be selected from any point that lies in the basin of attractor. In order to find the control parameter \(R\) that offers the maximum values of chaoticity and complexity, Fig.1 shows the bifurcation diagram of the peak of \(z\) (\(z\) max) versus the parameter \(R\). It is seen in Fig.1 that the system exhibits a period-doubling route to chaos. In addition, Fig.2 shows the plots of the positive LE versus the parameter \(R\). The chaoticity is a measure of the greatest LE, which is the average rate of growth of the distance between two nearby initial conditions that grows exponentially in time when averaged along the trajectory, leading to long-term unpredictability property.
The Lyapunov exponents can be employed for the estimation of the rate of entropy production and the fractal dimension commonly known as Kaplan-Yorke dimension $D_{KY}$ [18], i.e.

$$D_{KY} = j + \frac{1}{|LE_j|} \sum_{i=1}^{j} |LE_i| = k + \frac{LE_1 + LE_2}{|LE_1|}.$$  \hspace{1cm} (11)

where $k$ is a non-integer constant, and typically equals to 2 for three-dimensional chaotic systems. Table I summarizes chaotic regions for the parameter $R$ at the maximum LE ($R_{LE_{max}}$), the LEs and the corresponding $D_{KY}$. The system (1) has relatively wide chaotic range considered in five regions, including $0<R<14$, $15<R<36$, $45<R<65$, $67<R<92$, and $93<R<134$. Particularly, five chaotic regions yield higher values of both maximum LE and fractional $D_{KY}$ compared with the diffusive Lorenz system that possesses the maximum $D_{KY}$ of 2.2354 at $R=3.4693$ where the LEs are $(0.30791, 0, -1.30791)$. It is also seen from Table I that the maximum positive LE of 0.4299 and the maximum $D_{KY}$ of 2.3004 occur in the region $45<R<65$ where $R_{LE_{max}}$ is 60, which can be chosen as a demonstrating numerical value.

![Figure 1](image1.png)

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**Figure 1.** A bifurcation diagram exhibiting a period-doubling route to chaos of the peak of $z$ (‘z max’) versus the parameter $R$.

![Figure 2](image2.png)

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**Figure 2.** Plots of the maximum positive Lyapunov exponent versus the parameter $R$.  

B. Numerical Equilibria and Eigenvalues

The system (1) can be formulated using the specific value of $R$ at $R_{LE_{max}}=60$ as follows:

$$\begin{align*}
\dot{x} &= y - x - xy + |y| \\
\dot{y} &= z - 60 + xy + |y| \\
\dot{z} &= -60 + xy + |y|
\end{align*}$$

With reference to the previous analyses in (4) to (10), Table II summarizes the numerical values of equilibrium points, the three eigenvalues of Jacobian matrices, and the corresponding types of equilibrium points at $R_{LE_{max}}=60$. As shown in Table II, the attractor orbits around the two equilibrium points $P^*(7.262, 7.262, 0)$ and $P^*(-8.262, -8.262, 0)$, corresponding to the two-scroll attractor with mostly symmetric shape. It is also evident that the two equilibrium points are saddle focus nodes as the eigenvalue $\lambda_3$ is negative real value and $\lambda_{23}$ are a pair of complex conjugate eigenvalues with positive real parts.

![Table 1](image3.png)

**Table I.** Summary of five chaotic regions, and the corresponding values of $R_{LE_{max}}$, LEs, and $D_{KY}$.

<table>
<thead>
<tr>
<th>Regions $R_{LE_{max}}$</th>
<th>LEs at $p=-1$</th>
<th>$D_{KY}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0&lt;R&lt;14$</td>
<td>$(0.3696, 0.0003, -1.3688)$</td>
<td>2.2697</td>
</tr>
<tr>
<td>$15&lt;R&lt;36$</td>
<td>$(0.4172, 0.0011, -1.4160)$</td>
<td>2.2938</td>
</tr>
<tr>
<td>$45&lt;R&lt;65$</td>
<td>$(0.4299, 0.0004, -1.4294)$</td>
<td>2.3004</td>
</tr>
<tr>
<td>$67&lt;R&lt;92$</td>
<td>$(0.4260, 0.0005, -1.4256)$</td>
<td>2.2984</td>
</tr>
<tr>
<td>$93&lt;R&lt;134$</td>
<td>$(0.4210, 0.0013, -1.4197)$</td>
<td>2.2956</td>
</tr>
</tbody>
</table>

![Table 2](image4.png)

**Table II.** Summary of equilibrium points and the corresponding eigenvalues of Jacobian matrices at $R_{LE_{max}}=60$.

<table>
<thead>
<tr>
<th>Equilibria $P^*$</th>
<th>Eigenvalues $\lambda_{1,2,3}$</th>
<th>Equilibrium Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(7.262, 7.262, 0)$</td>
<td>$\lambda_{1,2,3}=0.3667 \pm 3.0657$</td>
<td>Saddle Focus</td>
</tr>
<tr>
<td>$(-8.262, -8.262, 0)$</td>
<td>$\lambda_{1,2,3}=0.3667 \pm 3.0657$</td>
<td>Saddle Focus</td>
</tr>
</tbody>
</table>

V. Simulation Results

In the time domain, Fig. 3(a) shows apparently chaotic waveforms whilst an apparently continuous broadband spectrum log|$x$| in the frequency domain is shown in Fig. 3(b). It can be seen from Fig. 3 that the system exhibits chaotic behaviors. By using the Fourth-order Runge-Kutta method to solve the system (12) with time step size of 0.001, the chaotic attractors are displayed in Figs. 4(a)–(d) for a three-dimensional view, an $x$–$y$ phase plane, an $x$–$z$ phase plane and a $y$–$z$ phase plane, respectively. It is seen in Fig. 4(a) that the attractor of three-dimensional view remains confined to the positive half-space of the $z$-axis. In addition, the attractors apparently exhibit a two-scroll topology that orbits around the two equilibrium points $P^*(7.262, 7.262, 0)$ and $P^*(-8.262, -8.262, 0)$.

Figs. 5(a), (b) and (c) visualize the Poincaré maps in planes where $x = 0$, $y = 0$ and $z = 0$, respectively. Several sheets of the attractors are displayed. It is noticeable that the Poincaré maps of many chaotic systems such as in the generalized Lorenz system [19] only show a branch.
with several twigs. The Poincaré maps in Figs. 5(a) and (b), however, consist of virtually symmetrical branches and a number of nearly symmetrical wings. In Fig. 5(c), it is clear that some sheets are folded.

VI. FORMING MECHANISMS OF COMPOUND STRUCTURES

Compound structures [20] generally demonstrate a forming mechanism of attractor, clarifying its topological structure. The compound structures of the system (12) may be demonstrated using a half-image operation to obtain only the left or the right half-image attractors, both of which can then be merged together as a compound structure. Such an operation can be revealed through the use of a controlled system of the following form

\[
\begin{align*}
\dot{x} &= y - x \\
\dot{y} &= -\tanh(x)z + u \\
\dot{z} &= -60 + xy + |x|
\end{align*}
\] (13)

where \( u \) is a control parameter. When \( u = -15 \), a left half-image of the original attractor shown in Fig. 4(d) can be isolated as illustrated in Fig. 6(a) for the \( y-z \) phase plane. In contrast, when \( u = 15 \), another right half-image of the original attractor can be isolated as illustrated in Fig. 6(b) for the \( y-z \) phase plane. It is evident in both left-image and right-image of the attractor in Fig. 6 that the system (13) consists of compound structures, which ultimately form the two-scroll attractor topology as previously depicted in Fig. 4(d).
Fig. 7 demonstrates gradual development of forming mechanisms of such compound structures. Different dynamical behaviors can be summarized as follows:

(a) When $|u| \geq 36$, the system (13) has limit cycles. For example, Fig. 7(a) shows a limit cycle at $u = 40$.

(b) When $23 \leq |u| \leq 36$, the system (13) also has limit cycles, forming different characteristics. For example, Fig. 7(b) shows a limit cycle at $u = 30$.

(c) When $6 \leq |u| \leq 23$, the system (13) demonstrates period-doubling bifurcations. For example, Fig. 7(c) shows such period-doubling bifurcations at $u = 14$.

(d) When $|u| \leq 6$, the system (13) exhibits a partially complete attractor as shown in Fig. 7(d) at $u = 5$.

VII. COMPARISONS OF CHAOTICITY AND COMPLEXITY

Table 3 summarizes the chaoticity and complexity between this paper and related references. For chaoticity, the divergence of flow $p$ typically affects the values of LEs and consequently sets the chaoticity measured by the positive $LE_{\text{max}}$. It is evident from Table 1 that this paper possesses the highest $LE_{\text{max}}$ of 0.4299 compared to the systems [9] and [12] in which $p$ is optimally simple at -1. Although the smaller values of $p$ in systems [1], [10], [12], [17] and [21] result in a higher values of $LE_{\text{max}}$, the negative LE is also decreased correspondingly, retaining the constant $D_{\text{KY}}$ as described in (11).

For complexity, this paper has offered the highest value of $D_{\text{KY}}=2.3004$. It is seen from Table 3 that the use of only piecewise-linear nonlinearity in [12] offers a relatively low $D_{\text{KY}}$ while the use of two nonlinearity in [9], [10], and [20] offer higher $D_{\text{KY}}$. In addition, the systems [15] and [17] require at least three quadratic nonlinearities with higher number of terms in ODEs, but the $D_{\text{KY}}$ is relatively low. The proposed chaotic system has attempted for more complex attractors through the interaction of two quadratic and two piecewise-linear nonlinearities, resulting in higher values of both chaoticity and complexity. Furthermore, circuit implementations for tanh and absolute value functions are relatively simple based on a single saturated amplifier and a signum nonlinearity circuit [18]. Therefore, the proposed system offers a potential alternative to highly complex chaotic system, but the possible implementation is simple.

![Figure 5. Poincaré maps in planes where (a) $x = 0$, (b) $y = 0$, (c) $z = 0$.](image-url)
CONCLUSIONS

The new chaotic system has been presented with a single parameter and a two-scroll attractor. The system has employed two quadratic nonlinearities and two piecewise-linear nonlinearities with only six-terms in ODEs. The system achieved high chaoticity of $0.429$ and high complexity of $2.3004$. Dynamic properties have been described in terms of symmetry, a dissipative system, an existence of attractor, equilibria, Jacobian matrix, bifurcations, Poincaré maps, waveforms, spectrum, and forming mechanisms of compound structures. This work offers a potential alternative to chaotic systems in applications of chaos to control and communication systems.

REFERENCES


Wimol San-Um was born in Nan Province, Thailand in 1981. He received BEng Degree in Electrical Engineering and MSc Degree in Telecommunications in 2003 and 2006, respectively, from Sirindhorn International Institute of Technology, Thammasat University, Thailand. In 2007, he was a research student at University of Applied Science Ravensburg-Weingarten, Germany. He received PhD in LSI Designs in 2010 from the Department of Electronic and Photonics System Engineering, Koc University of Technology, Japan. He is a lecturer at Computer Engineering Program, Faculty of Engineering, Thai-Nich Institute of Technology. His areas of research interests are analog integrated circuit designs, involving chaotic oscillators and switched-current circuits, and on-chip testing design, involving DFT and BIST techniques.

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