Numerical Results of Nonlinear Filtering Problem from Yau-Yau Method

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Abstract—In the paper, we introduce a kind of method to solve the nonlinear filtering problem. Firstly, we review the basic filtering problem and the reduction from robust Duncan-Mortensen-Zakai equation to Kolmogorov equation. Then we use the difference discrete method to solve the Kolmogorov equation. The result is given to prove that the solution of the difference scheme converges pointwise to the solution of the initial-value problem of the Kolmogorov equation. At last, the numerical results show that the numerical method can give the exact result.

Index Terms—Nonlinear filtering equation, DMZ equation, Kolmogorov equation

I. INTRODUCTION

One of the key goals of data assimilation is to obtain, recursively in time, good estimates of the state of a stochastic dynamical system based on noisy partial observations of the same. The major breakthrough in this classical problem of signal analysis was the landmark work of Kalman and Bucy [1]. The power of Kalman-Bucy filtering, which can be gauged from the broad spectrum of disciplines where it is used, lies in its elegant simplicity. Since then, the Kalman-Bucy filtering has proved useful in many areas such as navigational and guidance systems, radar tracking, solar mapping, and satellite orbit determination. Ever since the technique of the Kalman-Bucy filtering was popularized, there has been an intense interest in developing nonlinear filtering theory. In the 1960s and early 1970s, the basic approach to nonlinear filtering theory was via innovation methods originally proposed by Kailath [2] and Frost and Kailath[3] and subsequently rigorous developed by Fujisaki, Kallianpur, and Kunita [4] in 1972. But the difficulty with this approach is that innovation process is not, in general, explicitly computable. In the late 1970s, Brockett and Clark [5], Brockett [6] and Mitter [7] proposed the idea of using estimation algebras to construct a finite-dimensional nonlinear filtering. Yau and his coworkers have finally classified all finite dimensional estimation algebras of Maximal rank ([8]-[19]).

Although it is an interesting and challenging problem to classify all finite dimensional filtering, it appears that from Yau’s previous works, finite dimensional filtering simply does not exist for many practical situations. In ([20-21]), Yau and Yau proved the existence and decay estimates of the solution to the DMZ(Duncan-Mortensen-Zakai) equation under the assumption that the drift terms of the signal dynamic and observation dynamic respectively have linear growths. Later they [22] showed that the real-time solution of DMZ equation can be reduced to off-time solution of Kolmogorov equation if the drift terms have linear growths. Recently Yau and Yau ([23-24]) finally proved that the real-time solution of DMZ equation can be reduced to off-time solution of Kolmororov equation if the growth of the drift term of observation dynamic at infinity is faster than the growth of the drift term of signal dynamic at infinity.

In section 2, we recall the basic filtering problem. In section 3, we review the reduction from robust DMZ equation to Kolmogorov equation of Yau-Yau method. In section 4, we shall use finite difference method to derive the solution of Kolmororov equation and the numerical simulation of Yau-Yau method is shown.

II. INTRODUCTION OF NONLINEAR FILTERING PROBLEM

The filtering problem considered here is based on the signal observation model

\[ dx(t) = f(x(t))dt + g(x(t))dv(t) , \quad x(0) = x_0 , \]

\[ dy(t) = h(x(t))dt + dw(t) , \quad y(0) = 0 , \]

in which \( x, v, y \) and \( w \) are \( R^n, R^p, R^m \) and \( R^r \) valued processes and \( v, w \) have components that are independent,
standard Brownian processes. We further assume that \( n = p \cdot f \cdot g \) and \( h \) are vector-valued, matrix-valued and vector-valued \( C^\infty \) smooth functions. We shall refer to \( x(t) \) as the state of the system and \( y(t) \) as the observation at time \( t \).

Let \( \rho(x,t) \) denote the conditional probability density of the state given the observation \( \{ y(s) : 0 \leq s \leq t \} \). It is well known that \( \rho(x,t) \) is given by normalizing a function \( \sigma(t,x) \) that satisfies the following Duncan-Mortensen-Zakai equation

\[
d\sigma(t,x) = L_0 \sigma(t,x) dt + \sum_{i = 1}^{n} L_i \sigma(t,x) dy_i(t),
\]

\[
\sigma(0,x) = \sigma_0(x),
\]

(3)

Where

\[
L_0 = \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i} f_i \frac{\partial}{\partial x_i} - \sum_{i,j} \frac{\partial f_i}{\partial x_j} \frac{1}{2} \sum_{i} h_i^2.
\]

(4)

And for \( i = 1, 2, \ldots, m \), \( L_i \) is the zero degree differential operator of multiplication by \( h_i \), \( \sigma_0 \) is the probability density of the initial point.

Equation (3) is a stochastic partial differential equation. In real application, we are interested in constructing robust state estimators from observed sample paths with some property of robustness. From the robust algorithms proposed by Davis [25], it define a new unnormalized density

\[
u(t,x) = \exp(-\sum_{i=1}^{n} h_i(x) y_i(t)) \sigma(t,x).
\]

(5)

Then we can reduce the equation (3) to the following equation, which is called robust DMZ equation,

\[
\frac{\partial u}{\partial t}(t,x) + \frac{1}{2} \Delta u(t,x) + F(t,x) \cdot \nabla u + V(t,x) u(t,x) = \sigma_0(x),
\]

where drift function \( F(t,x) = -f(x) + \nabla k(t,x) \), source function

\[
V(t,x) = -\text{div}(f(x)) - \frac{1}{2} |h(x)|^2 + \frac{1}{2} \Delta k(t,x) - f(x) \cdot \nabla k(t,x) + \frac{1}{2} |\nabla k(t,x)|^2.
\]

III. YAU-YAU METHODS FOR THE ROBUST DMZ EQUATION

The fundamental problem of nonlinear filtering theory is how to solving the robust DMZ equation(6) in real time and in memoryless manner. In the section, we introduce the Yau-Yau algorithm which achieves this goal for a large class of filtering system with arbitrary initial distribution by reducing it to solve Kolmogorov equation.

Suppose that \( u(t,x) \) is the solution of robust D-M-Z equation and we want to compute \( u(t,x) \). Let \( T_i = \{ 0 = t_0 < t_1 < \cdots < t_i = \tau \} \) be a partition of \( [0, \tau] \). And let \( u_i(t,x) \) be a solution of the following partial differential equation for \( \tau_{i-1} \leq t \leq \tau_i \),

\[
\frac{\partial u_i}{\partial t}(t,x) = \frac{1}{2} \Delta u_i(t,x) + F(t_i,x) \cdot \nabla u_i + V(t_i,x) u_i,
\]

\[ u_i(\tau_{i-1},x) = u_{i-1}(\tau_{i-1},x). \]

(7)

(8)

Theorem 1. [23-24] Let \( u(t,x) \) and \( u_i(t,x) \) be the solutions of (6) and (7) respectively. For any given \( \varepsilon > 0 \) and sufficiently large \( n \),

\[
|u(t,x) - u_i(t,x)| \leq \varepsilon T e^{-\tau},
\]

(9)

and

\[
u(t,x) = \begin{cases} u_i(t,x) & \text{uniformly in } x \\
\end{cases}
\]

(10)

Where \( C \) is the constant and \( |T_i| = \sup_{\text{inside} \{ \tau_i - \tau_{i-1} \} } \) . So the solution of (7) approximate the solution of robust D-M-Z equation very well in the pointwise sense.

Therefore it remains to describe an algorithm to compute \( u_i(\tau_{i},x) \), which is based on the following property.

Proposition 1. [23-24] \( \tilde{u}(t,x) \) satisfies the following Kolmogorov equation

\[
\frac{\partial \tilde{u}}{\partial t}(t,x) = \frac{1}{2} \Delta \tilde{u}(t,x) - \sum_{i = 1}^{n} f_i(x) \frac{\partial \tilde{u}}{\partial x_i}(t,x)
\]

\[
- \left( \sum_{i = 1}^{n} \frac{\partial f_i}{\partial x_i}(x) + \frac{1}{2} \sum_{i} h_i^2(x) \right) \tilde{u}(t,x),
\]

for \( \tau_{i-1} \leq t \leq \tau_i \), if and only if

\[
u(t,x) = \exp(-\sum_{i=1}^{n} h_i(x) y_i(t)) \tilde{u}(t,x)
\]

satisfies the robust DMZ equation with observation being fixed at \( y(t_i) \),

\[
\frac{\partial u}{\partial t}(t,x) = \frac{1}{2} \Delta u(t,x) + F(t,x) \cdot \nabla u + V(t,x) u(t,x).
\]

(12)

From the property, \( u_i(\tau_{i},x) \) can be computed by \( \tilde{u}_i(\tau_{i},x) \) where \( \tilde{u}_i(\tau_{i},x) \) satisfies the following Kolmogorov equation

\[
\frac{\partial \tilde{u}_i}{\partial t}(t,x) = \frac{1}{2} \Delta \tilde{u}_i(t,x) - \sum_{i = 1}^{n} f_i(x) \frac{\partial \tilde{u}_i}{\partial x_i}(t,x)
\]

\[
- \left( \sum_{i = 1}^{n} \frac{\partial f_i}{\partial x_i}(x) + \frac{1}{2} \sum_{i} h_i^2(x) \right) \tilde{u}_i(t,x),
\]

\[
\tilde{u}_i(0,x) = \sigma_0(x) \exp\left( \sum_{i=1}^{n} h_i(x) y_i(t_i) \right).
\]

(13)

In fact,

\[
u_i(\tau_{i},x) = \exp\left( \sum_{i=1}^{n} h_i(x) y_i(t_i) \right) \tilde{u}_i(\tau_{i},x).
\]

(14)

For \( i \geq 2 \), \( u_i(\tau_{i},x) \) can be computed by \( \tilde{u}_i(\tau_{i},x) \) where \( \tilde{u}_i(\tau_{i},x) \) for \( \tau_{i-1} \leq t \leq \tau_i \) satisfies the following Kolmogorov equation

\[
\frac{\partial \tilde{u}_i}{\partial t}(t,x) = \frac{1}{2} \Delta \tilde{u}_i(t,x) - \sum_{i = 1}^{n} f_i(x) \frac{\partial \tilde{u}_i}{\partial x_i}(t,x)
\]

\[
- \left( \sum_{i = 1}^{n} \frac{\partial f_i}{\partial x_i}(x) + \frac{1}{2} \sum_{i} h_i^2(x) \right) \tilde{u}_i(t,x),
\]

\[
\tilde{u}_i(\tau_{i-1},x) = \exp\left( \sum_{i=1}^{n} h_i(x) y_i(t_i) \right) - y_i(\tau_{i-1}) \tilde{u}_{i-1}(\tau_{i-1},x).
\]

(15)
Observe that in the algorithm at step i, we only need the observation at time \( r_i \) and \( r_{i+1} \). We do not need any other previous observation data. Observe also that the Kolmogorov equation (15) is uniform for all time steps and it depends on observation \( y(t) \) only initial condition.

IV. NUMERICAL METHOD FOR THE KOLMOGOROV EQUATION

Consider the Kolmogorov equation (11), we use finite difference method to approximate the solution of the equation. We take the Kolmogorov equation

\[
\frac{\partial u}{\partial t} (t, x) = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (t, x) + p(x) \frac{\partial u}{\partial x} (t, x) + q(x) u(t, x),
\]

as an example to explain the method, where

\[
p(x) = -f(x), \quad q(x) = -(\frac{\partial f}{\partial x}(x) + \sum_{i=1}^{n} k_i^2(x)).
\]

To approximate the equation (16) by finite difference, we discretize the spatial and time domain by placing a grid over the domain. For convenience, we use a uniform grid, with grid spacing \( \Delta x \) and \( \Delta t \). The space-time domain of our problem then is approximated by the lattice of points. Notationally, we will denote \( u \) to be a function at the point \((k\Delta x, n\Delta t)\). To arrive at the finite difference equation used to approximate the Kolmogorov equation, we use the approximation

\[
u_k(n\Delta t, k\Delta x) \approx v_i^n - v_i^{n-1},
\]

\[
u_i(n\Delta t, k\Delta x) \approx v_{i+1}^n - 2v_i^n + v_{i-1}^n, \quad \frac{\Delta}{\Delta t}, \quad \Delta x
\]

\[
u_i(n\Delta t, k\Delta x) \approx v_i^n - v_i^{n-1}, \quad \frac{\Delta}{\Delta x}
\]

We see that

\[
u_i^n - nu_i^n - p(x)u_i^n - q(x)u
\]

\[
u_{i+1}^n - u_i^n - 2v_i^n + u_{i-1}^n
\]

\[
u_i^n - u_i^n - 2v_i^n + u_{i-1}^n
\]

\[
\frac{\partial}{\partial x}
\]

\[
\frac{\partial f}{\partial x}(x) + \sum_{i=1}^{n} k_i^2(x)
\]

where the above expression assumes that the higher order derivatives of \( u \) at \((k\Delta x, n\Delta t)\) are bounded.

Then we have the following theorem.

**Theorem 2.** Suppose \( q(x_i) < 0 \), and

\[
1 - \frac{2\Delta t}{\Delta x^2} \frac{\partial}{\partial x} (x_i \Delta t) + \Delta t q(x_i) \Delta t q(x_i) \geq 0,
\]

\[
\frac{\Delta t}{\Delta x^2} \frac{\partial}{\partial x} (x_i \Delta t) \geq 0,
\]

then the solution of the difference scheme

\[
v_{i+1} = v_i^n + \frac{\Delta t}{\Delta x^2} (v_i^n - 2v_i^n + v_{i-1}^n)
\]

\[
+ \frac{\Delta t}{\Delta x} f(x_i) (v_i^n - v_{i-1}^n) + \Delta t q(x_i) v_i^n,
\]

\[
\frac{\Delta t}{\Delta x} f(x_i) (v_i^n - v_{i-1}^n) + \Delta t q(x_i) v_i^n,
\]

convergences pointwise to the solution of the initial-value problem

\[
\frac{\partial u}{\partial t} (t, x) = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (t, x) + p(x) \frac{\partial u}{\partial x} (t, x) + q(x) u(t, x).
\]

Proof. We denote the exact solution to the initial-value problem by \( u = u(t, x) \) and set

\[
z_k^n = u(k\Delta x, n\Delta t) - v_k^n.
\]

If we insert \( u \) into equation (20), and multiply through by \( \Delta t \), we see that \( u_k^n = u(k\Delta x, n\Delta t) \) satisfies

\[
u_{k+1}^n = \nu_k^n + \frac{\Delta t}{\Delta x} (\nu_{k+1}^n - 2
\]

\[
\nu_k^n + \frac{\Delta t}{\Delta x} (\nu_{k+1}^n - 2v_k^n + v_{k-1}^n)
\]

\[
+ \frac{\Delta t}{\Delta x} f(x_i) (v_i^n - v_{i-1}^n) + \Delta t q(x_i) u_i^n
\]

\[
+ \Delta t^2 \Delta t q(x_i) u_i^n
\]

Then by subtracting equation (26) from equation (23), we see that \( z_k^n \) satisfies

\[
z_{k+1}^n = (1 - \frac{1}{\Delta t} + \frac{\Delta t}{2\Delta x^2} p(x_i) + \Delta t q(x_i))z_k^n
\]

\[
+ \frac{\Delta t}{2} \frac{\partial f}{\partial x}(x_i) (z_k^n)
\]

\[
+ \frac{\Delta t}{\Delta x} f(x_i) (z_{k+1}^n - z_{k-1}^n) + O(\Delta t^2 + \Delta t \Delta x).
\]

In order to obtain bounds for \( z_k^n \), we need to ensure that the coefficients of the three terms on the right of this equation are all nonnegative and have a sum no greater than unity. We are therefore led to the condition

\[
1 - \frac{2\Delta t}{2\Delta x^2} \frac{\partial}{\partial x} (x_i \Delta t) + \Delta t q(x_i) \Delta t q(x_i) \geq 0,
\]

\[
\frac{\Delta t}{\Delta x^2} \frac{\partial}{\partial x} (x_i \Delta t) \geq 0,
\]

as well as \( q(x_i) < 0 \). Then we have

\[
|z_{k+1}^n| \leq 1 - \frac{2\Delta t}{2\Delta x^2} \frac{\partial}{\partial x} (x_i \Delta t) + \Delta t q(x_i) |z_k^n| + \Delta t |z_{k+1}^n| + \Delta t \Delta x |z_{k+1}^n|
\]

\[
+ \frac{\Delta t}{\Delta x} f(x_i) (z_{k+1}^n - z_{k-1}^n) + A(\Delta t^2 + \Delta t \Delta x)
\]

\[
\leq z_k^* + A(\Delta t^2 + \Delta t \Delta x),
\]

where \( A \) is the constant associated with the "\( O^n \)" terms and depends on the assumed bound of the higher order derivatives of \( u \), and \( Z^* = \sup_k \{|z_k^n|\} \).

Then taking the sup over \( k \) on the left side of the above equation yields

\[
Z_{k+1}^* \leq Z_k^* + A(\Delta t^2 + \Delta t \Delta x).
\]

Applying the above equation repeatedly yields

\[
Z_{k+1}^* \leq Z_k^* + A(\Delta t^2 + \Delta t \Delta x) \leq \cdots
\]

\[
\leq Z_0^* + (n+1)A(\Delta t^2 + \Delta t \Delta x)
\]

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Since $Z^n = 0$, \[ |u(k \Delta x, (n+1) \Delta t) - v^n| \leq Z^{n+1}, \]
and $(n+1) \Delta t \to t$, then we have
\[ |u(k \Delta x, (n+1) \Delta t) - v^n| \leq (n+1) A(\Delta t^2 + \Delta x \Delta t) \to 0, \]
as $\Delta t, \Delta x \to 0$. □

A. Numerical simulation

Let $f(x) = \cos(x), h(x) = \sin(x)$, and $f(x) = x^2$, $h(x) = \sin(x)$ respectively, we can have figure 1 and 2, where $\sigma = \exp(-10 \times x^2)$. The dashed line denotes the mean of the conditional probability density $\rho$ and the solid line is generated from discrete extended Kalman filter equations. Figure 3 and 4 show the numerical behavior of the conditional probability density $\rho$ in different time.

Let $f_1(x, y) = -x^2 - y^2,$

\[ f_2(x, y) = x^2 + y^2, \]
\[ h_1(x, y) = x - y, \]
\[ h_2(x, y) = x + y, \]
and $f_1(x, y) = -\cos(x) - \sin(y),$
\[ f_2(x, y) = \cos(x) + \sin(y), \]
\[ h_1(x, y) = x - y, \]
\[ h_2(x, y) = x + y, \]
$\sigma = \exp(-5(x^2 + y^2))$, we have figure (5,6), in which figure(5,6) are the means about the $x, y$ respectively. The dashed line denotes the mean of the conditional probability density $\rho$ and the solid line is generated from discrete extended Kalman filter equations. Figure (7,8) show the numerical behavior of the conditional probability density $\rho$ in different time.

Fig. 1  

Fig. 2  

Fig. 3  

Fig. 4
V. CONCLUSION

In the paper, we use the Yau-Yau method to solve the nonlinear filtering problem, which reduce the robust DMZ equation to Kolmogorov equation. We give a finite difference schemes to derive the numerical solution of the Kolmogorov equation. A proof is given to prove that the solution of the difference scheme convergences pointwise to the solution of the initial-value problem of the Kolmogorov equation. At last, we give some numerical experiments to test the method.

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