$(\in,\in\vee q_{(\lambda,\mu)})$-Fuzzy Regular Subrings

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***Abstract*—Our aim in this paper is to introduce and study the new type of fuzzy subrings of a ring called fuzzy (resp. completely, weakly) regular subring, generalized fuzzy (resp. completely , weakly) regular subring, $(\in,\in\vee q_{(\lambda,\mu)})$ - fuzzy (resp. completely , weakly) regular subring and the direct products of them .Some of their algebraic properties are characterized by their level sets and extension principle. Finally, we give the properties of homomorphic image and homomorphic preimage of them.**

***Index Terms*— $(\in,\in\vee q_{(\lambda,\mu)})$- fuzzy subrings; generalized fuzzy subrings; homomorphic image; homomorphic preimage; level set; direct product**

I. Introduction

After the concept of fuzzy sets first introduced by Zadeh in 1965 [1], the theory of fuzzy mathematics has been widely used in Mathematics and many other areas. In 1971, Rosenfeld [2] introduced the concept of fuzzy subgroup, which spreads the area of fuzzy algebra. Since then, many scholars have published a series of papers [3-6]. Fuzzy algebra plays an important role in the field of computer, such as fuzzy codes , fuzzy prefix codes , maximal fuzzy prefix codes , fuzzy finite state machines , regular fuzzy languages and codes and so on [7] , so it's necessary for us to do further research on fuzzy algebra theory.

In 1981, W. J .Liu introduced the concept of fuzzy subring [8]. In 1992, Indian scholar S. K. Bhakat and P. Das introduced the concept of (α,β) -fuzzy subgroup [9] by using "belong to ($\in$)" relation and "quais-coincident with(q)"relation between the fuzzy point and the fuzzy sets . In fact, it's an important and useful generalization of Rosenfeld's fuzzy subgroup . In 1996, S. K. Bhakat and P .Das introduced the concept of (α,β) -fuzzy subring in [10], B . J .Zhao and H. G .Chi has introduced the concept of (λ,μ) -generalized fuzzy subring in [11]. Z.H.Liao et al extended Rosenfeld's fuzzy algebra, Bhakat and Das's fuzzy algebra and $(\bar{\in},\bar{\in}\vee\bar{q})$ -fuzzy algebra to $(\in,\in\vee q_{(\lambda,\mu)})$- fuzzy algebra, and do series of researches [12-15]. This paper is the continuation of the above work .we give the definition of fuzzy (resp. completely, weakly) regular subrings, generalized fuzzy (resp, completely, weakly) regular subrings, $(\in,\in\vee q_{(\lambda,\mu)})$ - fuzzy (resp. completely , weakly) regular subrings and the direct products of them .Besides some of the algebraic properties of them are characterized by their level sets and extension principle . Moreover , we also give the properties of homomorphic image and homomorphic preimage of them.

II. Preliminaries

From now on, R and H denote rings.

Definition 2.1 [15] If $x\in R$ and there exists $y\in R$, y is said to be group inverse of x if $xyx=x;\ yxy=y;$ $xy=yx$ .Be denoted by $x^{\#}$.

It is easy to proof that if there exists group inverse of x then it is unique and $x=(x^{\#})^{\#}$.

Definition 2.2[14] Let α , λ , $\mu\in[0,1]$ and $\lambda<\mu$, if $A(x)\ge\alpha$ then a fuzzy point x_α belongs to a fuzzy subset A , denoted by $x_\alpha\in A$; if $\lambda<\alpha$ and $A(x)+\alpha>2\mu$ then a fuzzy point x_α generalized quasicoincident with a fuzzy subset A , denoted by $x_\alpha q_{(\lambda,\mu)}A$. If $x_\alpha\in A$ or $x_\alpha q_{(\lambda,\mu)}A$, then $x_\alpha\in\vee q_{(\lambda,\mu)}A$.

Definition 2.3[3] we define stronger level set $A_{\alpha>}=\{x|A(x)>\alpha\}$.

Definition2.4 [3] Let $R_i(1\le i\le n)$ be rings and the direct product:

$$\prod_{1\le i\le n}R_i=\{(a_1,a_2,....a_n)|a_i\in R_i\}.$$

Then $\prod_{1\le i\le n}R_i$ is a ring under the operations as follows:

$(a_1,a_2,....a_n)+(b_1,b_2,...b_n)=(a_1+b_1,a_2+b_2,...a_n+b_n)$;

$(a_1,a_2,....a_n)(b_1,b_2,...b_n)=(a_1b_1,a_2b_2,...a_nb_n)$.

Definition 2.5 [3] Let $A_i(1\le i\le n)$ be fuzzy subsets of R_i , then a fuzzy subset $\prod_{1\le i\le n}A_i$ is defined as $(\prod_{1\le i\le n}A_i)(x_1,x_2,...x_n)$ = $\inf_{1\le i\le n}A_i(x_i)$ called fuzzy direct product.

Definition 2.6 [3] A fuzzy subset A is said to be a (λ,μ) -generalized fuzzy subring of R if for all $x,y\in R$, the following conditions are satisfied :

(1) $A(x+y)\vee\lambda\ge A(x)\wedge A(y)\wedge\mu$;

(2) $A(-x)\vee\lambda\ge A(x)\wedge\mu$;

(3) $A(xy)\vee\lambda\ge A(x)\wedge A(y)\wedge\mu$.

Theorem 2.1[6] Let A be a fuzzy subset of R. Then A is a generalized fuzzy subring of R if and only if $A_\alpha\ne\varnothing$ is a subring of R for all $\alpha\in(\lambda,\mu]$.

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Theorem 2.2 [6] Let A and B be generalized fuzzy subrings of a ring R ,then $A \bigcap B$ is a generalized fuzzy subring of R.

Theorem 2.3[6] A is a subring of R if and only if χ_A is a generalized fuzzy subring of R.

III. $(\in,\in \vee q_{(\lambda,\mu)})$ -Fuzzy (Completely) Regular Subrings

Let $x \in R$. Then we write $R_x = \{x' \mid x' \in R, xx'x = x\}$, $C_x = \{y \mid yx = xy, y \in R\}$.

Definition 3.1. A fuzzy subring A of a ring R is called a fuzzy regular subring of R if for all $x \in R$, there exists $x' \in R_x$ such that $A(x') \geq A(x)$.

Definition 3.2. A generalized fuzzy subring A of R is called a generalized fuzzy regular subring of R if for all $x \in R$, there exists $x' \in R_x$ such that $A(x') \vee \lambda \geq A(x) \wedge \mu$.

Definition 3.3. An $(\in,\in \vee q_{(\lambda,\mu)})$ -fuzzy subring A of a ring R is called a $(\in,\in \vee q_{(\lambda,\mu)})$ -fuzzy regular subring of R if for all $x \in R$ and $\alpha \in (\lambda,1]$, there exists $x' \in R_x$ such that $x_\alpha \in A$ implies $(x')_\alpha \in \vee q_{(\lambda,\mu)} A$.

Theorem 3.1. Let A be a fuzzy subset of ring R .Then the following properties are equivalent.

(1) $A(x') \vee \lambda \geq A(x) \wedge \mu$, for all $x \in R$ and exists $x' \in R_x$;

(2) For all $\alpha \in (\lambda,1]$, if $x_\alpha \in A$ then exists $x' \in R_x$ such that $(x')_\alpha \in \vee q_{(\lambda,\mu)} A$.

Proof. If (2)holds, then assume that there exists $x_0 \in R$ and for all $x_0' \in R_{x_0}$ such that $A(x_0') \vee \lambda < A(x_0) \wedge \mu$. Choose α such that $A(x_0') \vee \lambda < \alpha < A(x_0) \wedge \mu$ .Then $A(x_0) > \alpha$ and $\lambda < \alpha < \mu$, so $(x_0)_\alpha \in A$. We have $(x_0')_\alpha \in \vee q_{(\lambda,\mu)} A$. But $A(x_0') < \alpha < \mu$, therefore $A(x_0') + \alpha < \alpha + \alpha < 2\mu$, a contradiction. Thus (1) holds.

Conversely, for all $\alpha \in (\lambda, 1]$, $x_\alpha \in A$ then exists $x' \in R_x$ such that $A(x') \vee \lambda \geq A(x) \wedge \mu \geq \alpha \wedge \mu$. Then $A(x) \geq \alpha$. If $\alpha \leq \mu$ by $\lambda < \alpha$, then $A(x') \geq \alpha$. So $(x')_\alpha \in A$. If $\alpha > \mu$, then $A(x') \geq \mu$ .So $A(x') + \alpha \geq \alpha + \mu > 2\mu$ .Hence $(x')_\alpha \in \vee q_{(\lambda,\mu)} A$.

Theorem 3.2. Let A be a fuzzy subset of ring R .Then the following properties are equivalent.

(1) A is an $(\in,\in \vee q_{(\lambda,\mu)})$ -fuzzy regular subring of R;

(2) A is a generalized fuzzy regular subring of R;

(3) $A_\alpha \neq \varnothing$ is a regular subring of R for all $\alpha \in (\lambda,\mu]$.

Proof. From Theorem3.1,we know that (1)is equivalent to(2).

If (2) holds, based on Theorem2.1, it is sufficient to show that $A_\alpha \neq \varnothing$ is a subring of ring R for all $\alpha \in (\lambda,\mu]$. So we just need to proof that for all $\alpha \in (\lambda,\mu]$ and $x \in A_\alpha$, there exists $x' \in R_x$ such that $x' \in A_\alpha$ .In fact, by A is a generalized fuzzy regular subring of ring R , then there exists $x' \in R_x$ such that $A(x') \vee \lambda \geq A(x) \wedge \mu \geq \alpha \wedge \mu = \alpha$. For $\alpha \in (\lambda,\mu]$, So $A(x') \geq \alpha$,i.e., $x' \in A_\alpha$. Hence $A_\alpha \neq \varnothing$ is a regular subring of R.

Conversely,if (3) holds. Based on Theorem2.1, it is sufficient to show that A is a generalized fuzzy subring of ring R. So we just proof for all $x \in R$, exists $x' \in R_x$ such that $A(x') \vee \lambda \geq A(x) \wedge \mu$ .Assume that exists $x_0 \in R$ and for all $x_0' \in R_{x_0}$, $A(x_0') \vee \lambda < A(x_0) \wedge \mu$ .Choose α shch that $A(x_0') \vee \lambda < \alpha < A(x_0) \wedge \mu$ .Then $A(x_0) > \alpha$, $A(x_0') < \alpha$,i.e., $x_0 \in A_\alpha$, By $A_\alpha \neq \varnothing$ is a regular subring of ring R and $x_0 \in A_\alpha$ .Then exists $x_0' \in R_{x_0}$ such that $x_0' \in A_\alpha$, a contradiction .Thus for all $x \in R$, there exists $x' \in R_x$ such that $A(x') \vee \lambda \geq A(x) \wedge \mu$ .So(2)holds.

Theorem 3.3. Let a fuzzy subset A of R be a generalized fuzzy regular subring ,then $A_{\underline{\lambda}} = \{x \in R \,|\, A(x) > \lambda\}$ is a regular subring of R.

Proof. Since A is a generalized fuzzy regular subring of R ,then for every $x, y \in A_{\underline{\lambda}}$,we have $A(x+y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu > \lambda \wedge \mu = \lambda$, $A(-x) \vee \lambda \geq A(x) \wedge \mu > \lambda$ and $A(xy) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu > \lambda$,i.e., $A(x+y) > \lambda$, $A(-x) > \lambda$ and $A(xy) > \lambda$,so $x+y, xy, -x \in A_{\underline{\lambda}}$. Therefore $A_{\underline{\lambda}} = \{x \in R \,|\, A(x) > \lambda\}$ is a subring of R .Now for every $x \in A_{\underline{\lambda}}$ and exists $x' \in R_x$ such that $A(x') \vee \lambda \geq A(x) \wedge \mu > \lambda$ which implies $x' \in A_\lambda$ .Thus $A_{\underline{\lambda}}$ is a regular subring of R.

Theorem 3.4. Let A and B be generalized fuzzy regular subrings of R and H respectively, then $A \times B$ is a generalized fuzzy regular subring of $R \times H$.

Proof. Firstly, we prove that $A \times B$ is a generalized fuzzy subring of $R \times H$ .For all $(x_1,x_2),(y_1,y_2) \in R \times H$, A and B are generalized fuzzy regular subrings of R and H respectively, we have $(A \times B)((x_1,x_2)+(y_1,y_2)) \vee \lambda = (A \times B)((x_1+y_1, x_2+y_2)) \vee \lambda = (A(x_1+y_1) \wedge B(x_2+y_2)) \vee \lambda = (A(x_1+y_1) \vee \lambda) \wedge (B(x_2+y_2) \vee \lambda) \geq A(x_1) \wedge A(y_1) \wedge \mu \wedge B(x_2) \wedge B(y_2) \wedge \mu = (A \times B)((x_1,x_2)) \wedge (A \times B)((y_1,y_2)) \wedge \mu$. Similarly, we can prove that $(A \times B)(-(x_1,x_2)) \vee \lambda \geq (A \times B)((x_1,x_2)) \wedge \mu$ and $(A \times B)((x_1,x_2)(y_1,y_2)) \vee \lambda \geq (A \times B)((x_1,x_2)) \wedge (A \times B)((y_1,y_2)) \wedge \mu$ .So $A \times B$ is a generalized fuzzy subring of $R \times H$.

Also, since A and B are generalized fuzzy regular subrings of R and H respectively, for all $x\in R$ and $y\in H$,there exist $x'\in R_x$ and $y'\in R_y$ i.e., $(x',y')\in R_x\times R_y$ such that $A(x')\vee\lambda\geq A(x)\wedge\mu, A(y')\vee\lambda\geq A(y)\wedge\wedge\mu$. $(A\times B)((x',y'))\vee\lambda=(A(x')\wedge B(y'))\vee\lambda=(A(x')\vee\lambda)\wedge(B(y')\vee\lambda)\geq A(x)\wedge\mu\wedge B(y)\wedge\mu=(A\times B)((x,y))\wedge\mu$ .So $A\times B$ is a generalized fuzzy regular subring of $R\times H$.

Theorem 3.5 Let A_i be generalized fuzzy regular subrings of R_i , then $\prod_{1\leq i\leq n}A_i$ is a generalized fuzzy regular subring of $\prod_{1\leq i\leq n}R_i$.

Proof. Firstly ,we prove that $\prod_{1\leq i\leq n}A_i$ is a generalized fuzzy subring of $\prod_{1\leq i\leq n}R_i$ .For all $x,y\in\prod_{1\leq i\leq n}R_i$, $x=(x_1,x_2,...x_n)$ and $y=(y_1,y_2,...y_n)$,then $(\prod_{1\leq i\leq n}A_i)(x+y)\vee\lambda=\inf A_i(x_i+y_i)\vee\lambda=\inf(A_i(x_i+y_i)\vee\lambda)\geq\inf(A_i(x_i)\wedge A_i(y_i)\wedge\mu)=\inf A_i(x_i)\wedge\inf A_i(y_i)\wedge\mu=(\prod_{1\leq i\leq n}A_i)(x)\wedge(\prod_{1\leq i\leq n}A_i)(y)\wedge\mu$ .Similarly, $(\prod_{1\leq i\leq n}A_i)(-x)\vee\lambda=\inf A_i(-x_i)\vee\lambda=\inf(A_i(-x_i)\vee\lambda)\geq\inf(A_i(x_i)\wedge\mu)=\inf A_i(x_i)\wedge\mu=(\prod_{1\leq i\leq n}A_i)(x)\wedge\mu$. We can prove $\prod_{1\leq i\leq n}A_i(xy)\vee\lambda\geq(\prod_{1\leq i\leq n}A_i)(x)\wedge(\prod_{1\leq i\leq n}A_i)(y)\wedge\mu$ using the same method.

Therefore $\prod_{1\leq i\leq n}A_i$ is a generalized fuzzy subring of $\prod_{1\leq i\leq n}R_i$.

Now for every $x\in\prod_{1\leq i\leq n}R_i$ and $x=(x_1,x_2,...x_n)$,since A_i is generalized fuzzy regular subrings of R_i ,so there exists $x_i'\in R_{x_i}$ such that $A_i(x_i')\vee\lambda\geq A_i(x_i)\wedge\mu$. In fact ,exists $x'=(x_1',x_2',...x_n')\in\prod_{1\leq i\leq n}R_{x_i}$ such that $(\prod_{1\leq i\leq n}A_i)(x')\vee\lambda=\inf A_i(x_i')\vee\lambda=\inf(A_i(x_i')\vee\lambda)\geq\inf(A_i(x_i)\wedge\mu)=\inf A_i(x_i)\wedge\mu=(\prod_{1\leq i\leq n}A_i)(x)\wedge\mu$ .So $\prod_{1\leq i\leq n}A_i$ is a generalized fuzzy regular subring of $\prod_{1\leq i\leq n}R_i$.

Definition 3.4. A fuzzy subring A of a ring R is called a fuzzy completely regular subring of R if for all $x\in R$, there exists $x^*\in R_x\cap C_x$ such that $A(x^*)\geq A(x)$

Definition 3.5. A generalized fuzzy subring A of a ring R is called a generalized fuzzy completely regular subring of R if for all $x\in R$,there exist $x^*\in R_x\cap C_x$ such that $A(x^*)\vee\lambda\geq A(x)\wedge\mu$.

Definition 3.6. An $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy subring A of R is called an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy completely regular subring of R if for all $x\in R$ and $\alpha\in(\lambda,1]$, there exists $x^*\in R_x\cap C_x$ such that $x_\alpha\in A$ implies $(x_0^*)_\alpha\in\vee q_{(\lambda,\mu)}A$.

Theorem 3.6. Let A be a fuzzy subring of ring R .Then the following properties are equivalent.

(1) For all $x\in R$, there exist $x^*\in R_x\cap C_x$ such that $A(x^*)\vee\lambda\geq A(x)\wedge\mu$;

(2) For all $x\in R$ and $\alpha\in(\lambda,1]$, there exists $x^*\in R_x\cap C_x$ such that $x_\alpha\in A$ implies $(x_0^*)_\alpha\in\vee q_{(\lambda,\mu)}A$;

(3) for all $x\in R$,there exists $x^{\#}\in R$ such that $A(x^{\#})\vee\lambda\geq A(x)\wedge\mu$,

Proof. (2) $\Rightarrow$(1) Assume that there exists $x_0\in R$ and for all $x_0^*\in R_{x_0}\cap C_{x_0}$ such that $A(x_0^*)\vee\lambda<A(x_0)\wedge\mu$. Choose α such that $A(x_0^*)\vee\lambda<\alpha<A(x_0)\wedge\mu$ .Then $A(x_0)>\alpha$ and $\lambda<\alpha<\mu$, so $(x_0)_\alpha\in A$, then $(x_0^*)_\alpha\in\vee q_{(\lambda,\mu)}A$. But $A(x_0^*)<\alpha<\mu$, therefore $A(x_0^*)+\alpha<\alpha+\alpha<2\mu$, a contradiction .So for all $x\in R$, there exist $x^*\in R_x\cap C_x$ such that $A(x^*)\vee\lambda\geq A(x)\wedge\mu$.

(1) $\Rightarrow$ (2) For all $\alpha\in(\lambda,1]$ and $x_\alpha\in A$ there exists $x^*\in R_x\cap C_x$ such that $A(x^*)\vee\lambda\geq A(x)\wedge\mu\geq\alpha\wedge\mu$. If $\alpha\leq\mu$, by $\lambda<\alpha$, then $A(x^*)\geq\alpha$. So $(x^*)_\alpha\in A$. If $\alpha>\mu$, then $A(x^*)\geq\mu$ .So $A(x^*)+\alpha\geq\alpha+\mu>2\mu$ .So $(x^*)_\alpha q_{(\lambda,\mu)}A$. Hence $(x^*)_\alpha\in\vee q_{(\lambda,\mu)}A$.

(3) $\Rightarrow$(1) It is easy to prove.

(1) $\Rightarrow$(3) For all f $x\in R$, there exists $x^*\in R_x\cap C_x$ such that $A(x^*)\vee\lambda\geq A(x)\wedge\mu$ Since $xx^*x=x, xx^*=x^*x$, then exists $x^{\#}=x^*xx^*$ such that $A(xx^*)\vee\lambda\geq(A(x)\wedge A(x^*))\vee\lambda=(A(x)\vee\lambda)\wedge(A(x^*)\vee\lambda)\geq A(x)\wedge\mu$ and $A(x^{\#})\vee\lambda=A(x^*xx^*)\vee\lambda\geq(A(x^*)\wedge A(xx^*))\vee\lambda=(A(x^*)\vee\lambda)\wedge(A(xx^*)\vee\lambda)\geq A(x)\wedge\mu$. Thus for all $x\in R$,there exists $x^{\#}$ such that $A(x^{\#})\vee\lambda\geq A(x)\wedge\mu$.

Theorem 3.7. Let A is a fuzzy subset of ring R .Then the following properties are equivalent.

(1) A is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy completely regular subring of ring R;

(2) A is a generalized fuzzy completely regular subring of ring R ;

(3) A is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy subring of ring R,and for all $\alpha\in(\lambda,1]$, if $x_\alpha\in A$ then $(x^{\#})_\alpha\in\vee q_{(\lambda,\mu)}A$.

Proof. Straightforward.

Theorem 3.8. Let $\{A_i\}_{i\in I}$ be generalized fuzzy completely regular subrings of R and for all $i,j\in I, A_i\subseteq A_j$ or $A_j\subseteq A_i$ .Then $\bigcup_{i\in I}A_i$ is a generalized fuzzy completely regular subring of R .

Proof. Firstly , we prove that $\vee_{i\in I}(A_i(x)\wedge A_i(y)\wedge\mu)=(\bigcup_{i\in I}A_i)(x)\wedge(\bigcup_{i\in I}A_i)(y)\wedge\mu$.

Obviously , $\vee_{i\in I}(A_i(x)\wedge A_i(y)\wedge\mu)\leq(\bigcup_{i\in I}A_i)(x)\wedge(\bigcup_{i\in I}A_i)($

$y)\wedge\mu$ .Assume that $\vee_{i\in I}(A_i(x)\wedge A_i(y)\wedge\mu)\neq(\bigcup_{i\in I}A_i)(x)$ $\wedge(\bigcup_{i\in I}A_i)(y)\wedge\mu$, then there exists r such that $\vee_{i\in I}(A_i(x)$ $\wedge A_i(y)\wedge\mu)<r<(\bigcup_{i\in I}A_i)(x)\wedge(\bigcup_{i\in I}A_i)(y)\wedge\mu$ .Since $A_i\subseteq A_j$ or $A_j\subseteq A_i$ for all $i,j\in I$,then exists $k\in I$ such that $r<A_k(x)\wedge A_k(y)\wedge\mu$ .On the other hand , $A_i(x)\wedge A_i(y)\wedge\mu<r$ for all $i\in I$,a contradiction . Therefore $\vee_{i\in I}(A_i(x)\wedge A_i(y)\wedge\mu)=(\bigcup_{i\in I}A_i)(x)\wedge(\bigcup_{i\in I}A_i)(y)\wedge\mu$.

Now for all $x,y\in R$,

(1) $(\bigcup_{i\in I}A_i)(x+y)\vee\lambda=\vee_{i\in I}A_i(x+y)\vee\lambda=\vee_{i\in I}(A_i(x+y)\vee\lambda)\geq\vee_{i\in I}(A_i(x)\wedge A_i(y)\wedge\mu)=(\bigcup_{i\in I}A_i)(x)\wedge(\bigcup_{i\in I}A_i)(y)\wedge\mu$;

(2) $(\bigcup_{i\in I}A_i)(-x)\vee\lambda=\vee_{i\in I}A_i(-x)\vee\lambda=\vee_{i\in I}(A_i(-x)\vee\lambda)\geq\vee_{i\in I}(A_i(x)\wedge\mu)=(\bigcup_{i\in I}A_i)(x)\wedge\mu$;

(3) $(\bigcup_{i\in I}A_i)(xy)\vee\lambda=\vee_{i\in I}A_i(xy)\vee\lambda=\vee_{i\in I}(A_i(xy)\vee\lambda)\geq\vee_{i\in I}(A_i(x)\wedge A_i(y)\wedge\mu)=(\bigcup_{i\in I}A_i)(x)\wedge(\bigcup_{i\in I}A_i)(y)\wedge\mu$;

(4)For all $x\in R$,there exists $x^{\#}\in R$ such that $(\bigcup_{i\in I}A_i)(x^{\#})\vee\lambda=\vee_{i\in I}A_i(x^{\#})\vee\lambda=\vee_{i\in I}(A_i(x^{\#})\vee\lambda)\geq\vee_{i\in I}(A_i(x)\wedge\mu)=(\bigcup_{i\in I}A_i)(x)\wedge\mu$.

Thus $\bigcup_{i\in I}A_i$ is a generalized fuzzy completely regular subring of R .

Theorem 3.9. A fuzzy subset A of a ring R is a generalized fuzzy completely regular subring of ring R if and only if $A_\alpha\neq\varnothing$ is a completely regular subring of ring R for all $\alpha\in(\lambda,\mu]$.

Proof. Based on Theorem2.1 and Theorem3.1 it is sufficient to show that $A_\alpha\neq\varnothing$ is a subring of ring R for all $\alpha\in(\lambda,\mu]$. So we just proof for all $\alpha\in(\lambda,\mu]$ and $x\in A_\alpha$,there exists $x^{\#}$ such that $x^{\#}\in A_\alpha$ .In fact, by A is a generalized fuzzy completely regular subring of R , then there exists $x^{\#}\in R$ such that $A(x^{\#})\vee\lambda\geq A(x)\wedge\mu\geq\alpha\wedge\mu=\alpha$. For $\alpha\in(\lambda,\mu]$, then $A(x^{\#})\geq\alpha$, that is to say $x^{\#}\in A_\alpha$. Hence $A_\alpha\neq\varnothing$ is a completely regular subring of R.

Conversely,Based on Theorem2.1 and Theorem3.1, it is sufficient to show that A is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy subring of ring R. So we just need to proof that for all $x\in R$, there exists $x^{\#}\in R$ such that $A(x^{\#})\vee\lambda\geq A(x)\wedge\mu$ .Assume that there exists $x_0\in R$ such that $A(x_0{}^{\#})\vee\lambda<A(x_0)\wedge\mu$ .Choose α such that $A(x_0{}^{\#})\vee\lambda<\alpha<A(x_0)\wedge\mu$ .Then $A(x_0)>\alpha$, $A(x_0{}^{\#})<\alpha$,i.e., $x_0\in A_\alpha$, By $A_\alpha\neq\varnothing$ is a completely regular subring of R and $x_0\in A_\alpha$ .Then exists $x_0{}^{\#}$ such that $x_0{}^{\#}\in A_\alpha$, a contradiction. Thus A is a generalized fuzzy completely regular subring of R .

Theorem 3.10. Let a fuzzy subset A of R be a generalized fuzzy completely regular subring ,then we obtain that $A_{\underline{\lambda}}=\{x\mid A(x)>\lambda\}$ is a completely regular subring of R .

Proof $A_{\underline{\lambda}}=\{x\in R\mid A(x)>\lambda\}$ is a subring of R based on theorem 3.3 . For every $x\in A_{\underline{\lambda}}$,there exists $x^{\#}$ such that $A(x^{\#})\vee\lambda\geq A(x)\wedge\mu>\lambda$ which implies $x^{\#}\in A_{\underline{\lambda}}$. Thus $A_{\underline{\lambda}}$ is a completely regular subring of R .

Theorem 3.11. Let A and B be generalized fuzzy completely regular subrings of R and H respectively, then $A\times B$ is a generalized fuzzy completely regular subring of $R\times H$.

Proof. we obtain that $A\times B$ is a generalized fuzzy subring of $R\times H$ based on theorem 3.4 .

Also , since A and B are generalized fuzzy completely regular subrings of R and H respectively , for all $x\in R$ and $y\in H$,there exist $x^{\#}\in R$ and $y^{\#}\in H$ i.e., $(x^{\#},y^{\#})\in R\times H$ such that $A(x^{\#})\vee\lambda\geq A(x)\wedge\mu, A(y^{\#})\vee\lambda\geq A(y)\wedge\wedge\mu$,so $(A\times B)((x^{\#},y^{\#}))\vee\lambda=(A(x)^{\#}\wedge B(y^{\#}))\vee\lambda=(A(x^{\#})\vee\lambda)\wedge(B(y^{\#})\vee\lambda)\geq A(x)\wedge\mu\wedge B(y)\wedge\mu=(A\times B)((x,y))\wedge\mu$ .So $A\times B$ is a generalized fuzzy completely regular subring of $R\times H$.

Theorem 3.12 Let A_i be generalized fuzzy completely regular subrings of R_i , then $\prod_{1\leq i\leq n}A_i$ is a generalized fuzzy completely regular subring of $\prod_{1\leq i\leq n}R_i$.

Proof. $\prod_{1\leq i\leq n}A_i$ is a generalized fuzzy subring of $\prod_{1\leq i\leq n}R_i$ based on theorem3.5.

Now for every $x\in\prod_{1\leq i\leq n}R_i$ and $x=(x_1,x_2,...x_n)$,since A_i is generalized fuzzy completely regular subrings of R_i ,so there exists $x_i{}^{\#}\in R_i$ such that $A_i(x_i{}^{\#})\vee\lambda\geq A_i(x_i)\wedge\mu.$ In fact ,there exists $x^{\#}=(x_1{}^{\#},x_2{}^{\#},...x_n{}^{\#})\in\prod_{1\leq i\leq n}R_i$ such that $(\prod_{1\leq i\leq n}A_i)(x^{\#})\vee\lambda=\inf A_i(x_i{}^{\#})\vee\lambda=\inf(A_i(x_i{}^{\#})\vee\lambda)\geq\inf(A_i(x_i)\wedge\mu)=\inf A_i(x_i)\wedge\mu=(\prod_{1\leq i\leq n}A_i)(x)\wedge\mu$ .So $\prod_{1\leq i\leq n}A_i$ is a generalized fuzzy completely regular subring of $\prod_{1\leq i\leq n}R_i$.

Theorem 3.13. Let $f:R\to H$ be a full homomorphism mapping , if A is an $(\in,\in\vee q_{(\lambda,\mu)})$ - fuzzy subring of R, then $f(A)$ is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy subring of H.

The proof is omitted.

Theorem 3.14. Let $f:R\to H$ be a homomorphism mapping , if B is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy subring of H, then $f^{-1}(B)$ is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy subring of R.

The proof is omitted.

Theorem 3.15. Let $f: R\to H$ be a full homomorphism mapping .If A is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy regular subring of R, then $f(A)$ is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy regular subring of H .

Proof Based on Theorem 3.13 , we know that $f(A)$ is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy subring of H.For all $z\in H$, exists $x\in f^{-1}(z)\in R$, Since A is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy regular subring of R, there exists $x'\in R_x$ such that $A(x')\vee\lambda\geq A(x)\wedge\mu$. For $f(x)f(x')f(x)=f(xx'x)=f(x)=z$,so $z'\triangleq f(x')\in R_z$,

$$\begin{aligned}
&f(A)(z)\wedge\mu\\
&=\sup\{A(x)\mid f(x)=z,x\in R\}\wedge\mu\\
&=\sup\{A(x)\wedge\mu\mid f(x)=z,x\in R\}\\
&\leq\sup\left\{A(x')\vee\lambda\mid f(x)=z,x'\in R_x\right\}\\
&=\sup\left\{A(x')\mid f(x)=z,x'\in R_x\right\}\vee\lambda\\
&\leq\sup\left\{A(x')\mid z'\triangleq f(x')\in R_z,x'\in R_x\right\}\vee\lambda\\
&\leq\sup\left\{A(x)\mid z'=f(x),x\in R,z'\in R_z\right\}\vee\lambda\\
&=f(A)(z')\vee\lambda
\end{aligned}$$

Where $z'\in R_z$.

So $f(A)$ is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy regular subring of H.

Theorem 3.16. Let $f: R\to H$ be a isomomorphism mapping.If B is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy regular subring of H , then $f^{-1}(B)$ is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy regular subring of R .

Proof Based on Theorem 3.14 , we know that $f^{-1}(B)$ is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy subring of R . For all $x\in R$, $z=f(x)\in H$, Since B is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy regular subring of H, therefore

$$\begin{aligned}
f^{-1}(B)(x)\wedge\mu&=B(f(x))\wedge\mu\\
&\leq B(z)\vee\lambda \qquad \textit{exists } z\in R_{f(x)}
\end{aligned}$$

By f is a surjection , so there exists $x'\in R$ such that $f(x')=z$,then $f(x)=f(x)zf(x)=f(x)f(x')f(x)=f(xx'x)$, since f is injection , then $xx'x=x$,therefore $x'\in R_x$ and

$$\begin{aligned}
f^{-1}(B)(x)\wedge\mu&\leq B(z)\vee\lambda\\
&=B(f(x'))\vee\lambda\\
&=f^{-1}(B)(x')\vee\lambda
\end{aligned}$$

So $f^{-1}(B)$ is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy regular subring of R .

Theorem 3.17. Let $f: R\to H$ be a full homomorphism mapping ,if A is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy completely regular subring of R, then $f(A)$ is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy completely regular subring of H.

Proof $f(A)$ is an $(\in,\in\vee q_{(\lambda,\mu)})$ - fuzzy subring of H based on theorem 3.13. For all $z\in H$, there exists $x\in R$ such that $f(x)=z$ .Since A is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy completely regular subring of R,then there exists $x^{\#}\in R$ such that $A(x^{\#})\vee\lambda\geq A(x)\wedge\mu$, therefore , exists $z^{\#}=f(x^{\#})\in H$ such that

$$\begin{aligned}
f(A)(z^{\#})\vee\lambda&=\sup\left\{A(x)\mid f(x)=z^{\#}\right\}\vee\lambda\\
&=\sup\left\{A(x)\vee\lambda\mid f(x)=z^{\#}\right\}\\
&\geq\sup\left\{A(x^{\#})\vee\lambda\mid f(x^{\#})=z^{\#}\right\}\\
&\geq\sup\{A(x)\wedge\mu\mid f(x)=z\}\\
&=\sup\{A(x)\mid f(x)=z\}\wedge\mu\\
&=f(A)(z)\wedge\mu
\end{aligned}$$

Therefore $f(A)$ is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy completely regular subring of H.

Theorem 3.18. Let $f: R\to H$ be homomorphism mapp-ing , if B is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy completely regular subring of H, then $f^{-1}(B)$ is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy completely regular subring of R.

Proof Based on Theorem 3.14 , we know that $f^{-1}(B)$ is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy subring of R. For all $x\in R$, $z=f(x)\in H$, for B is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy completely regular subring of H, then there exists $z^{\#}\in H$ such that $B(z^{\#})\vee\lambda\geq B(z)\wedge\mu$ .Therefore

$$\begin{aligned}
f^{-1}(B)(x^{\#})\vee\lambda&=B(f(x^{\#}))\vee\lambda\\
&=B(f(x)^{\#})\vee\lambda\\
&\geq B(f(x))\wedge\mu\\
&=f^{-1}(B)(x)\wedge\mu
\end{aligned}$$

Thus $f^{-1}(B)$ is an $(\in,\in\vee q_{(\lambda,\mu)})$ - fuzzy completely regular subring of R .

IV. $(\in,\in\vee q_{(\lambda,\mu)})$ -FUZZY WEAKLY REGULAR SUBRINGS

Definition 4.1 A generalized fuzzy subring A of a ring R is said to be a generalized fuzzy weakly regular subring if for all $x\in R$, $\bigvee_{x'\in R_x}A(x')\vee\lambda\geq A(x)\wedge\mu$.

Definition 4.2 An $(\in,\in\vee q_{(\lambda,\mu)})$ - fuzzy subring A of a ring R is said to be an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy weakly regular subring if for all $x\in R$ and $\alpha\in(\lambda,1]$, $x_\alpha\in A$ implies $(\bigcup_{x'\in R_x}x')_\alpha\in\vee q_{(\lambda,\mu)}A$.

Theorem 4.1. Let A be a fuzzy subset of a ring R .Then the following conditions are equivalent:

(1) A is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy weakly regular subring ;

(2) A is a generalized fuzzy weakly regular subring;

(3) $A_{\alpha>}\neq\Phi$ is a regular subring of R for all $\alpha\in[\lambda,\mu)$.

Proof $(1)\Rightarrow(2)$

By definition 4.1,we have A is a generalized fuzzy subring of R .Now

We only need to prove that $\underset{x'\in R_x}{\vee} A(x')\vee\lambda\geq A(x)\wedge\mu$ for all $x\in R$ .Assume that there exists $x_0\in R$ such that $\underset{x_0'\in R_{x_0}}{\vee} A(x_0')\vee\lambda<A(x_0)\wedge\mu$, choose α such that $\underset{x_0'\in R_{x_0}}{\vee} A(x_0')\vee\lambda<\alpha<A(x_0)\wedge\mu$ .Then $A(x_0)>\alpha$ and $\lambda<\alpha<\mu$,i.e., $(x_0)_\alpha\in A$,then $(\underset{x_0'\in R_{x_0}}{\cup} x_o')_\alpha\in\vee q_{(\lambda,\mu)}A$ based on definition 4.2 and $(x_0)_\alpha\in A$ .But $\underset{x_0'\in R_{x_0}}{\vee} A(x_0')<\alpha$,so $A(\underset{x_0'\in R_{x_0}}{\cup} x_o')+\alpha=\underset{x_0'\in R_{x_0}}{\vee} A(x_0')+\alpha<2\alpha<2\mu$,a contradiction.Thus (2)holds.

$(2)\Rightarrow(1)$

A is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy subring of R based on theorem 2.1 .Since A is a generalized fuzzy weakly regular subring of R and $(x)_\alpha\in A$,then $\underset{x'\in R_x}{\vee} A(x')\vee\lambda\geq A(x)\wedge\mu\geq\alpha\wedge\mu$ for all $x\in R$.

Case 1:

When $\alpha\leq\mu$,then $\underset{x'\in R_x}{\vee} A(x')\vee\lambda\geq\alpha$,by $\lambda<\alpha$ so $\underset{x'\in R_x}{\vee} A(x')\geq\alpha$,i.e., $A(\underset{x'\in R_x}{\cup} x')\geq\alpha$,that is $(\underset{x'\in R_x}{\cup} x')_\alpha\in A$.

Case 2:

When $\alpha>\mu$,then we have $\underset{x'\in R_x}{\vee} A(x')\vee\lambda\geq\mu$ which implies

$\underset{x'\in R_x}{\vee} A(x')=A(\underset{x'\in R_x}{\cup} x')\geq\mu$,therefore $A(\underset{x'\in R_x}{\cup} x')+\alpha\geq\mu+\alpha>2\mu$,so $(\underset{x'\in R_x}{\cup} x')_\alpha q_{(\lambda,\mu)}A$.

combine case 1 and case 2 ,we have $(\underset{x'\in R_x}{\cup} x')_\alpha\in\vee q_{(\lambda,\mu)}A$ .Thus A is an $(\in,\in\vee q_{(\lambda,\mu)})$ -fuzzy weakly regular subring.

$(2)\Rightarrow(3)$

Firstly,we prove that $A_{\alpha>}\neq\Phi$ is a subring of R for all $\alpha\in[\lambda,\mu)$. By A is a generalized fuzzy subring of R and for all $x,y\in A_{\alpha>}$,we have $A(x+y)\vee\lambda\geq A(x)\wedge A(y)\wedge\mu>\alpha$ .Since $\lambda\leq\alpha$,so $A(x+y)>\alpha$, i.e., $x+y\in A_{\alpha>}$ .Similarly,we can prove that $-x\in A_{\alpha>}$ and $xy\in A_{\alpha>}$ .Therefore $A_{\alpha>}\neq\Phi$ is a subring of R for all $\alpha\in[\lambda,\mu)$.

Now we prove that $A_{\alpha>}\neq\Phi$ is a regular subring of R. By $x\in A_{\alpha>}$ and A is a generalized fuzzy weakly regular subring,we have $\underset{x'\in R_x}{\vee} A(x')\vee\lambda\geq A(x)\wedge\mu>\alpha$ which implies $\underset{x'\in R_x}{\vee} A(x')>\alpha$ .we define $\underset{x'\in R_x}{\vee} A(x')-\alpha=\varepsilon$,then $\varepsilon>0$ and exists $x'\in R_x$ such that $A(x')>\underset{x'\in R_x}{\vee} A(x')-\varepsilon=\alpha$,i.e., $x'\in A_{\alpha>}$ .Thus $A_{\alpha>}$ is a regular subring of R for all $\alpha\in[\lambda,\mu)$.

$(3)\Rightarrow(2)$

Firstly,we prove that A is a generalized fuzzy subring of R.Assume that there exist $x_0,y_0\in R$ such that $A(x_0+y_0)\vee\lambda<\alpha<A(x_0)\wedge A(y_0)\wedge\mu$,then $A(x_0)>\alpha, A(y_0)>\alpha$ and $\lambda<\alpha<\mu$,i.e., $x_0,y_0\in A_{\alpha>}$,since $A_{\alpha>}$ is a subring of R for all $\alpha\in[\lambda,\mu)$,so we obtain that $x_0+y_0\in A_{\alpha>}$,that is to say $A(x_0+y_0)>\alpha$ which is a contradiction to $A(x_0+y_0)<\alpha$. Thus $A(x+y)\vee\lambda\geq A(x)\wedge A(y)\wedge\mu$ for all $x,y\in R$ .Simlarly , we can prove $A(-x)\vee\lambda\geq A(x)\wedge\mu$ and $A(xy)\vee\lambda\geq A(x)\wedge A(y)\wedge\mu$ .Therefore A is a generalized fuzzy subring of R.

Next, we prove that for all $x\in R$, $\underset{x'\in R_x}{\vee} A(x')\vee\lambda\geq A(x)\wedge\mu$ .Assume that there exists $x_0\in R$ such that $\underset{x_0'\in R_{x_0}}{\vee} A(x_0')\vee\lambda<\alpha<A(x_0)\wedge\mu$ then $A(x_0)>\alpha$, $\underset{x_0'\in R_{x_0}}{\vee} A(x_0')<\alpha$ and $\lambda<\alpha<\mu$,i.e., $x_0\in A_{\alpha>}$,by (3) we obtain that there exists $x_0'\in R_{x_0}$ such that $x_0'\in A_{\alpha>}$, then $A(x_0')>\alpha$,a contradiction.Thus A is a generalized fuzzy weakly regular subring of R.

Corollary4.1. Let A be a generalized fuzzy weakly regular subring ,then $A_{\underline{\lambda}}=\{x\,|\,A(x)>\lambda\}$ is a regular subring of R .

Theorem 4.2.A nonempty subset A is a regular subring of R if and only if χ_A is a generalized fuzzy weakly regular subring of R .

Proof Based on theorem 2.4 ,we know that χ_A is a generalized fuzzy subring of R .Assume that there exists $x_0\in R$ such that $\underset{x_0'\in R_{x_0}}{\vee}\chi_A(x_0')\vee\lambda<\chi_A(x_0)\wedge\mu$ .Choose α such that $\underset{x_0'\in R_{x_0}}{\vee}\chi_A(x_0')\vee\lambda<\alpha<\chi_A(x_0)\wedge\mu$,then $\chi_A(x_0)>\alpha$, $\underset{x_0'\in R_{x_0}}{\vee}\chi_A(x_0')<\alpha$ and $\lambda<\alpha<\mu$,i.e., $\chi_A(x_0)=1$ which implies that $x_0\in A$,by A is a regular subring of R ,so there exists $x_0'\in R_{x_0}$ such that $x_0'\in A$, i.e., $\chi_A(x_0')=1>\alpha$,a contradiction.Therefore χ_A is a generalized fuzzy weakly regular subring of R .

Conversely , from theorem2.4 ,we obtain that A is a subring of R .For all $x\in A$,since χ_A is a generalized fuzzy weakly regular subring of R ,so $\underset{x'\in R_x}{\vee}\chi_A(x')\vee\lambda\geq \chi_A(x)\wedge\mu=\mu$,by $\lambda<\mu$,then $\underset{x'\in R_x}{\vee}\chi_A(x')\geq\mu$ so $\underset{x'\in R_x}{\vee}\chi_A(x')=1$,i.e.,there exists $x'\in R_x$ such that $\chi_A(x')=1$,so $x'\in A$ .So A is a regular subring of a ring R .

Theorem 4.3 Let $f: R\to H$ be a full homomorphism mapping,if A is a generalized fuzzy weakly regular subring of R ,then $f(A)$ is a generalized fuzzy weakly regular subring of H .

Proof $f(A)$ is a generalized fuzzy subring of H based on Theorem 3.13 .For all $z\in H$ exists $x\in R$ such that $f(x)=z$,since A is a generalized fuzzy weakly regular subring of R ,then $\underset{x'\in R_x}{\vee}A(x')\vee\lambda\geq A(x)\wedge\mu$ and

$f(x)f(x')f(x)=f(xx'x)=f(x)=z$, then $f(x')\in R_Z$.

$$\begin{aligned}
&\underset{z'\in R_z}{\vee} f(A)(z')\vee\lambda\\
&=\underset{z'\in R_z}{\vee}\sup\{A(x)\mid f(x)=z'\}\vee\lambda\\
&=\sup\{\underset{z'\in R_z}{\vee}A(x)\mid f(x)=z'\}\vee\lambda\\
&\geq\sup\{\underset{x'\in R_x}{\vee}A(x')\}\vee\lambda\\
&=\sup\{\underset{x'\in R_x}{\vee}A(x')\vee\lambda\}\\
&\geq\sup\{A(x)\wedge\mu\}\\
&=\sup\{A(x)\}\wedge\mu\\
&\geq\sup\{A(x)\mid f(x)=z\}\wedge\mu\\
&=f(A)(z)\wedge\mu
\end{aligned}$$

Thus $f(A)$ is a generalized fuzzy weakly regular subring of H .

Theorem 4.4 Let $f:R\to H$ be a isomomorphism mapping , if B is a generalized fuzzy weakly regular subring of H ,then $f^{-1}(B)$ is a generalized fuzzy weakly regular subring of R .

Proof From Theorem3.14,we obtain that $f^{-1}(B)$ is a generalized fuzzy subring of R . For all $x\in R$, then $z=f(x)\in H$ .Since B is a generalized fuzzy weakly regular subring of H , then $\underset{z'\in R_z}{\vee}B(z')\vee\lambda\geq B(z)\wedge\mu$.

By $f:R\to H$ be a surjection, thus exists $y\in R$, such that $z'=f(y)$ and $f(x)z'f(x)=f(x)f(y)f(x)=f(xyx)=f(x)$,Since $f:R\to H$ be a injection , then $xyx=x$. Therefore we have $z'=f(y)\in R_Z$ and $y\in R_x$

$$\begin{aligned}
&\underset{x'\in R_x}{\vee} f^{-1}(B)(x')\vee\lambda\\
&=\underset{x'\in R_x}{\vee}B(f(x'))\vee\lambda\\
&\geq\underset{y\in R_x}{\vee}B(f(y))\vee\lambda\\
&=\underset{z'\in R_z}{\vee}B(z')\vee\lambda\\
&\geq B(z)\wedge\mu\\
&\geq B(f(x))\wedge\mu\\
&=f^{-1}(B)(x)\wedge\mu
\end{aligned}$$

So $f^{-1}(B)$ is a generalized fuzzy weakly regular subring of R .

V. CONCLUTIONS

In this paper , we gave the definitions of fuzzy (resp. completely, weakly) regular subrings ,generalized fuzzy (resp. completely,weakly) regular subrings, $(\in,\in\vee q_{(\lambda,\mu)})$- fuzzy (resp completely, weakly) regular subrings and direct products of them . Moreover, we also discussed the algebraic properties of them by their level sets and extension principle .Finally , the properties of their homomorphic image and homomorphic preimage were discussed. The applications in computers need to be researched further.

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