

Comparison of B-spline Method and Finite Difference Method to Solve BVP of Linear ODEs

Jincai Chang, Qianli Yang, Long Zhao
 College of Science, Hebei United University
 Tangshan Hebei 063009, China
 {jincai@heut.edu.cn}

Abstract—B-spline functions play important roles in both mathematics and engineering. To describe a numerical method for solving the boundary value problem of linear ODE with second-order by using B-spline. First, the cubic B-spline basis functions are introduced, then we use the linear combination of cubic B-spline basis to approximate the solution. Finally, we obtain the numerical solution by solving tri-diagonal equations. The results are compared with finite difference method through two examples which shows that the B-spline method is feasible and efficient.

Index Terms—B-spline function, Boundary-value problem, Finite difference method

I. INTRODUCTION

Ordinary Differential Equations (ODE) has a long history and widely applied in many fields. The numerical solution of ODE has made great development in the 20th century. There have been emerged many new ideas as well as many complex methods for solving ODE, so that the numerical methods for solving ODE has been deepened.

Systems of ordinary differential equation have been applied to many problems in physics, engineering, biology and so on. The theory of spline functions is a very active field of approximation theory and boundary value problems (BVPs) when numerical aspects are considered. In this paper, we discuss a direct method based on B-spline for two-point boundary value problems of second-order ordinary differential equation. There are many publications dealing with this problem with some methods. Reference [1] For instance, B-spline applied to dealing with the non-linear problems; Reference [2] A finite difference method has been proposed; Reference [3-6] In a series of paper by Caglar BVPs of third, fifth were solved using fourth and sixth-degree splines; B-spline method for solving linear system of second-order boundary value and singular boundary value problems etc.

In the present paper, a cubic B-spline is used to solve two-point boundary value problems as the following linear systems which are assumed to have a unique solution in the interval [0,1].

$$\begin{cases} y''(x) + m(x)y'(x) + n(x)y(x) = f(x), & 0 \leq x \leq 1 \\ y(0) = 0, & y(1) = 0 \end{cases} \quad (1)$$

where $m(x)$, $n(x)$ and $f(x)$ are given functions, and $m(x), n(x)$ are continuous.

In section 2 we have given the definition of the B-spline method. The spline technique presents to approximate the solution of two-point boundary value problems in section 3. In section 4 we have solved two problems using the method and the max-absolute errors and graphs have also been shown. Section 5, reports the major conclusion and further developments.

II. THE CUBIC B-SPLINE

A. The definition of the B-spline function

Reference [7] Let $\Omega = \{x_0, x_1, \dots, x_n\}$ be a set of partition of $[0,1]$, the zero degree B-spline is defined as follows:

$$B_{i,0} = \begin{cases} 1, & x \in [x_i, x_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

and for positive p , it is defined in the following recursive form:

$$B_{i,p}(x) = \frac{x - x_i}{x_{i+p} - x_i} B_{i,p-1}(x) + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} B_{i+1,p-1}(x), p \geq 2$$

We apply this recursion to get the cubic B-spline, it is defined as follows:

$$B_{0,3}(x) = \frac{1}{6h^3} \begin{cases} x^3, & x \in [0, h) \\ -3x^3 + 12hx^2 - 12h^2x + 4h^3, & x \in [h, 2h) \\ 3x^3 - 24hx^2 + 60h^2x - 44h^3, & x \in [2h, 3h) \\ -x^3 + 12hx^2 - 48h^2x + 64h^3, & x \in [3h, 4h) \\ 0, & \text{otherwise} \end{cases}$$

B. The properties of B-spline functions

(1) Translation Invariance:

$$B_{i-1,p}(x) = B_{0,p}(x - (i-1)h), i = -3, -2, \dots$$

(2) Compact Supported:

$$B_{i,p}(x) = 0, x \notin [x_i, x_{i+p+1})$$

(3) Derivation formula:

$$B_{i,p}^{(k)}(x) = \frac{p!}{(p-k)!} \sum_{j=0}^k \alpha_{k,j} B_{i+j,p-k}$$

Where

$$\begin{cases} \alpha_{0,0} = 1 \\ \alpha_{k,0} = \frac{\alpha_{k-1,0}}{x_{i+p-k+1} - x_i} \\ \alpha_{k,k} = \frac{-\alpha_{k-1,k-1}}{x_{i+p+1} - x_{i+k}} \\ \alpha_{k,j} = \frac{\alpha_{k-j,j} - \alpha_{k-1,j-1}}{x_{i+p+j-k+1} - x_{i+j}} \end{cases}$$

III. B-SPLINE SOLUTIONS FOR LINEAR BOUNDARY VALUE PROBLEMS

Let

$$y(x) = \sum_{j=-3}^{n-1} c_j B_{j,3}(x). \tag{2}$$

be an approximate solution of Eq.(1), where c_i is unknown real coefficient and $B_{j,3}(x)$ are cubic B-spline functions. Let x_0, x_1, \dots, x_n are $n+1$ grid points in the interval $[a, b]$, so that $x_i = a + ih$, $i = 0, 1, \dots, n$, $x_0 = a, x_n = b, h = (b-a)/n$. It is required that the approximate solution (2) satisfies the differential equation at the points $x = x_i$. Putting (3) in (1), it follows that

$$\sum_{j=-3}^{n-1} c_j [B_{j,3}''(x_i) + m(x_i)B_{j,3}'(x_i) + n(x_i)B_{j,3}(x_i)] = f(x_i), i = 0, 1, \dots, n \tag{3}$$

and boundary condition can be written as

$$\sum_{j=-3}^{n-1} c_j B_{j,3}(0) = 0, \text{ for } x = 0. \tag{4}$$

$$\sum_{j=-3}^{n-1} c_j B_{j,3}(1) = 0, \text{ for } x = 1. \tag{5}$$

The spline solution of (1) is obtained by solving the following matrix equation. Then, a system of $n+3$ linear equations in the $n+3$ unknowns $c_{-3}, c_{-2}, \dots, c_{n-1}$ are obtained, using (2) can obtain the numerical solution. This system can be written in the matrix-vector form as follows:

$$AB = F \tag{6}$$

Where

$$B = [c_{-3}, c_{-2}, \dots, c_{n-1}]^T, \\ F = [0, f(x_0), f(x_1), \dots, f(x_n), 0]^T$$

and A is an $(n+3) \times (n+3)$ -dimensional tri-diagonal matrix given by

$$A = \begin{bmatrix} 1 & 4 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ a_0(x_0) & b_0(x_0) & c_0(x_0) & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & a_1(x_1) & b_1(x_1) & c_1(x_1) & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & a_n(x_n) & b_n(x_n) & c_n(x_n) \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 4 & 1 \end{bmatrix} \tag{7}$$

also the coefficients in the matrix A have the following form

$$a_i(x_i) = \frac{6}{h^2} + m(x_i) \frac{-3}{h} + n(x_i), (i = 0, 1, \dots, n) \\ b_i(x_i) = \frac{-12}{h^2} + 4n(x_i), (i = 0, 1, \dots, n) \\ c_i(x_i) = \frac{6}{h^2} + m(x_i) \frac{3}{h} + n(x_i), (i = 0, 1, \dots, n)$$

Then, a system of linear equations can be built as shown below:

$$\begin{bmatrix} 1 & 4 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ a_0(x_0) & b_0(x_0) & c_0(x_0) & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & a_1(x_1) & b_1(x_1) & c_1(x_1) & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & a_n(x_n) & b_n(x_n) & c_n(x_n) \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} c_{-3} \\ c_{-2} \\ \vdots \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{bmatrix} = 6 \begin{bmatrix} 0 \\ f(x_0) \\ \vdots \\ \vdots \\ f(x_n) \\ 0 \end{bmatrix}$$

IV. NUMERICAL RESULTS

Reference [8] In this section, two numerical examples are studied by B-spline, finite difference method. The results obtained by the method are compared with the analytical solution, so that we get the maximum absolute errors, then demonstrate the accuracy of the B-spline method. We can find that our method in comparison with the method of finite difference is much better with a view to accuracy and utilization. Moreover, the maximum absolute errors are given in Table 1 and 2. The numerical results are illustrated in Fig. 1, 2, 3 and Fig. 4.

Example 1: Solve the following boundary value problem

$$\begin{cases} y''(x) - y'(x) = -e^{x-1} - 1, & 0 \leq x \leq 1 \\ y(0) = 0, & y(1) = 0 \end{cases} \tag{8}$$

The analytical solution:

$$y(x) = x(1 - e^{x-1})$$

Respectively, the observed maximum absolute errors for various value of n are given in Table 1. The numerical results are illustrated in Fig.1 and 2.

Method 1: we can get the coefficient matrix A by using(7) for $n = 10$

$$A = \begin{bmatrix} 1 & 4 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ (6+3h)/h^2 & -12/h^2 & (6-3h)/h^2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & (6+3h)/h^2 & -12/h^2 & (6-3h)/h^2 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & (6+3h)/h^2 & -12/h^2 & (6-3h)/h^2 \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 4 & 1 \end{bmatrix}$$

And

$$F = [0, -8.2073, -8.4394, \dots, -11.4290, -12.0000]^T$$

Then, if $h = 0.1$, we can find B as follows:

$$B = [-0.0657, 0.0012, 0.0608, \dots, 0.0050, -0.1102]^T$$

And we can get the function

$$y(x) = \sum_{j=-3}^{n-1} c_j B_{j,3}(x)$$

for example

$$y_0(x) = -0.2x^3 - 0.365x^2 + 0.6325x - 0.0000166667$$

$$x \in [0, 0.1];$$

$$y_1(x) = -0.233x^3 - 0.355x^2 + 0.6315x + 0.000016$$

$$x \in [0.1, 0.2];$$

$$y_2(x) = -0.25x^3 - 0.3449x^2 + 0.6294x + 0.00015,$$

$$x \in [0.2, 0.3];$$

$$y_3(x) = -0.3x^3 - 0.3x^2 + 0.616x + 0.0015,$$

$$x \in [0.3, 0.4];$$

$$y_4(x) = -0.32x^3 - 0.28x^2 + 0.608x + 0.002483,$$

$$x \in [0.4, 0.5];$$

$$y_5(x) = -0.4x^3 - 0.155x^2 + 0.5455x + 0.0129833,$$

$$x \in [0.5, 0.6];$$

$$y_6(x) = -0.417x^3 - 0.125x^2 + 0.5275x + 0.0165833,$$

$$x \in [0.6, 0.7];$$

$$y_7(x) = -0.467x^3 - 0.02x^2 + 0.454x + 0.033733,$$

$$x \in [0.7, 0.8];$$

$$y_8(x) = -0.6x^3 + 0.3x^2 + 0.198x + 0.102,$$

$$x \in [0.8, 0.9];$$

$$y_9(x) = -0.6x^3 + 0.3x^2 + 0.1979x + 0.102,$$

$$x \in [0.9, 1];$$

Therefore, numerical solutions are obtained by the B-spline method, they follow that

$$y_1 = 0.0593827,$$

$$y_2 = 0.110234,$$

$$y_3 = 0.1512$$

$$y_4 = 0.1806167,$$

$$y_5 = 0.1969833,$$

$$y_6 = 0.198083334$$

$$y_7 = 0.18165523,$$

$$y_8 = 0.1452,$$

$$y_9 = 0.08571$$

the analytical solutions are given by

$$y_1(x_1) = 0.0593,$$

$$y_2(x_2) = 0.1101,$$

$$y_3(x_3) = 0.1510,$$

$$y_4(x_4) = 0.1805,$$

$$y_5(x_5) = 0.1967,$$

$$y_6(x_6) = 0.1978,$$

$$y_7(x_7) = 0.1814,$$

$$y_8(x_8) = 0.1450,$$

$$y_9(x_9) = 0.0856,$$

and

$$y_1(x_1) - y_1 = -0.0000827,$$

$$y_2(x_2) - y_2 = -0.000134,$$

$$y_3(x_3) - y_3 = -0.0002,$$

$$y_4(x_4) - y_4 = 0.000117,$$

$$y_5(x_5) - y_5 = -0.0002833,$$

$$y_6(x_6) - y_6 = 0.00023334$$

$$y_7(x_7) - y_7 = -0.00025523,$$

$$y_8(x_8) - y_8 = -0.0002,$$

$$y_9(x_9) - y_9 = -0.00011,$$

Thus, the max-absolute error is given by

$$\delta = 0.00025523$$

Reference [9] Method 2 Finite difference method: At first, the interval of solution is divided into many small regions and get the set of internal node. In these nodes, we use difference coefficient instead of differential. We reject the truncation error and establish the differential equations. Then, we can obtain the numerical solution by combining the boundary conditions.

Consider the linear boundary value problem

$$y'' = q_1(x)y' + q_2(x)y + q_3(x), x \in [a, b]. \quad (9)$$

$$A = \begin{bmatrix} 1 & 4 & 1 & 0 & 0 \dots & 0 & 0 & 0 \\ \phi h^2 + \phi h - 2 & -12h - 8 & \phi h^2 - \phi h - 2 & 0 & 0 \dots & 0 & 0 & 0 \\ 0 & \phi h^2 + \phi h - 2 & -12h - 8 & \phi h^2 - \phi h - 2 & 0 \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \dots & \phi h^2 + \phi h - 2 & -12h - 8 & \phi h^2 - \phi h - 2 \\ 0 & 0 & 0 & 0 & 0 \dots & 1 & 4 & 1 \end{bmatrix}$$

And

$$F = [0, -12, -12, \dots, -12, -12, 0]^T$$

Then, if $h = 0.1$, we can find B as follows:

$$B = [-0.0638, 0.0013, \dots, 0.0071, -0.1273]^T$$

And we can get the function $y(x) = \sum_{j=3}^{n-1} c_j B_{j,3}(x)$, for

example

$$y_0(x) = -0.1x^3 - 0.385x^2 + 0.6125x + 0.0000166667$$

$$x \in [0, 0.1];$$

$$y_1(x) = -0.083x^3 - 0.39x^2 + 0.613x,$$

$$x \in [0.1, 0.2];$$

$$y_2(x) = -0.217x^3 - 0.31x^2 + 0.597x - 0.0010666667$$

$$x \in [0.2, 0.3];$$

$$y_3(x) = -0.25x^3 + 29.255x^2 - 8.2725x + 0.0020,$$

$$x \in [0.3, 0.4];$$

$$y_4(x) = -0.35x^3 - 0.16x^2 + 0.54x + 0.0084,$$

$$x \in [0.4, 0.5];$$

$$y_5(x) = -0.5x^3 + 0.065x^2 + 0.4275x + 0.027,$$

$$x \in [0.5, 0.6];$$

$$y_6(x) = -0.67x^3 + 0.365x^2 + 0.2475x + 0.063,$$

$$x \in [0.6, 0.7];$$

$$y_7(x) = -0.9x^3 + 0.855x^2 - 0.0955x + 0.14315,$$

$$x \in [0.7, 0.8];$$

$$y_8(x) = -1.183x^3 + 1.535x^2 - 0.6395x + 0.289,$$

$$x \in [0.8, 0.9];$$

$$y_9(x) = -1.57x^3 + 2.57x^2 - 1.57x + 0.568,$$

$$x \in [0.9, 1];$$

Therefore, numerical solutions are obtained by the B-spline method, they follow that

$$y_1 = 0.05657,$$

$$y_2 = 0.1042973333,$$

$$y_3 = 0.1464166667$$

$$y_4 = 0.1763666667,$$

$$y_5 = 0.1939916667,$$

$$y_6 = 0.1982966667$$

$$y_7 = 0.18655,$$

$$y_8 = 0.1537706667,$$

$$y_9 = 0.0909366667,$$

the analytical solutions are given by

$$y_1(x_1) = 0.0572,$$

$$y_2(x_2) = 0.1061,$$

$$y_3(x_3) = 0.1460$$

$$y_4(x_4) = 0.1758,$$

$$y_5(x_5) = 0.1940,$$

$$y_6(x_6) = 0.1983$$

$$y_7(x_7) = 0.1858,$$

$$y_8(x_8) = 0.1524,$$

$$y_9(x_9) = 0.0928$$

and

$$y_1(x_1) - y_1 = 0.00063,$$

$$y_2(x_2) - y_2 = 0.0018,$$

$$y_3(x_3) - y_3 = -0.00042,$$

$$y_4(x_4) - y_4 = -0.00057,$$

$$y_5(x_5) - y_5 = -0.0000083,$$

$$y_6(x_6) - y_6 = 0.0000033,$$

$$y_7(x_7) - y_7 = -0.00075,$$

$$y_8(x_8) - y_8 = -0.0014,$$

$$y_9(x_9) - y_9 = 0.001863,$$

Thus, the max-absolute error is given by

$$\delta = 0.001863$$

Method 2: The numerical solutions are obtained by the Finite difference method, they follow that

- $y_1=0.0399,$
- $y_2=0.0897,$
- $y_3= 0.1302,$
- $y_4=0.1604,$
- $y_5=0.1787,$
- $y_6=0.1827,$
- $y_7=0.1695,$
- $y_8=0.1350,$
- $y_9=0.0735,$

also the max-absolute error is given by $\delta = 0.0193$

From the results, we will see the difference between them and conclude that the B-spline method is the better to interpolate any smooth functions than others. The numerical results for our example are shown in Table 1 and 2, which show that there is a big difference for the errors between B-spline method and the Finite difference method unless there is no remarkable difference among the accuracy of the other method in the case where f is sufficiently smooth.

TABLE I

Shows the max-absolute errors for the two methods with respect to the true solution

| Methods | H | Max-absolute errors |
|--------------------------|-----|---------------------|
| Finite difference method | 0.1 | 0.0004 |
| B-spline method | 0.1 | 0.00025523 |

TABLE II

Shows the max-absolute errors for the two methods with respect to the true solution

| Methods | H | Max-absolute errors |
|--------------------------|-----|---------------------|
| Finite difference method | 0.1 | 0.0193 |
| B-spline method | 0.1 | 0.001863 |

V. CONCLUSION AND OUTLOOK

A family of B-spline method has been considered for the numerical solution of boundary value problems of linear ordinary differential equations. The cubic B-spline has been tested on a problem. From the test examples, we can say that the accuracy is better than the finite difference method. The numerical results showed that the present method is an applicable technique and approximates the solution very well. The implementation of the present method is a very easy, acceptable, and valid scheme. This method gives comparable results and is easy to compute. Also this method produces a spline function which may be used to obtain the solution at any point in the range, whereas the finite difference method gives the solution only at the chosen knots. This method is easily tractable and can readily be applied to other

problems of differential equations. Several references given in this paper are of great practical importance but space constraints did not allow their discussion here. Finally, it can be observed from this article that a significant amount of work has been done and there is a large scope of work to be done in this field.

Reference [12-21] The above two examples are the deformation of singular perturbation problem. The singularly-perturbed differential equation is that

$$-\epsilon y''(x) + m(x)y'(x) + n(x)y(x) = f(x)$$

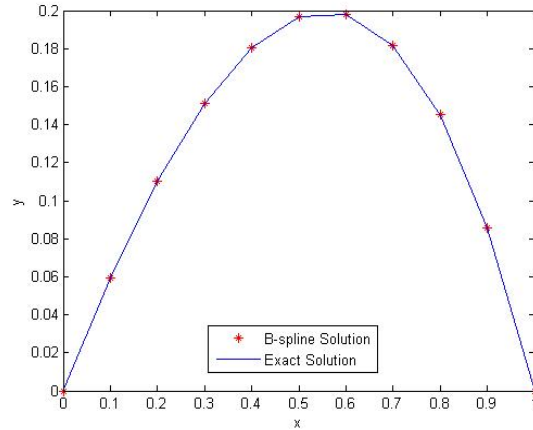


Fig1. B-spline

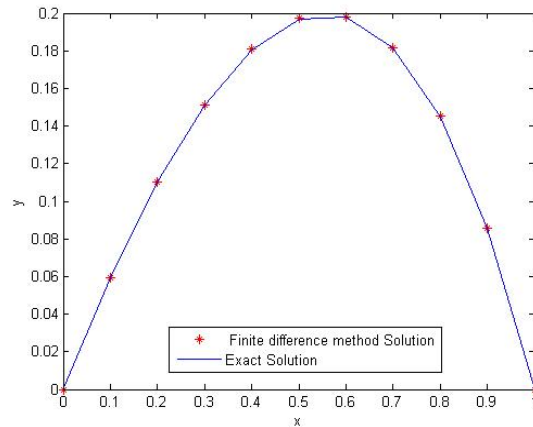


Fig2. Finite Difference

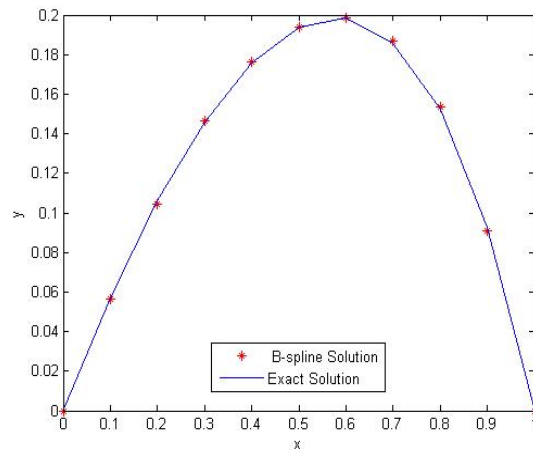


Fig3. B-spline

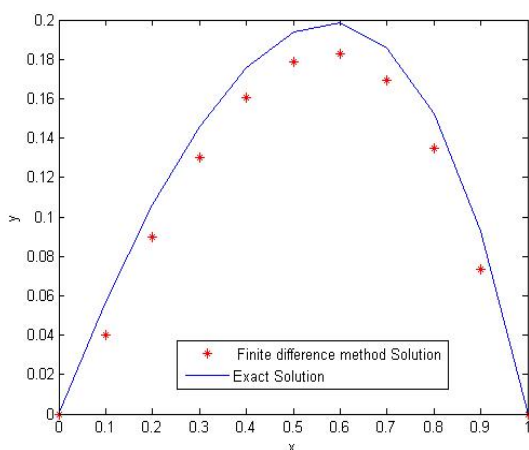


Fig4. Finite Difference

subject to $y(0) = A$ and $y(1) = B$ where $0 < \varepsilon \leq 1$, ε is a positive parameter, $m(x)$ and $n(x)$ are sufficiently smooth real valued functions. It is so attractive to mathematicians due to the fact that the solution exhibits a multi-scale character, that is, regions of rapid change in the solution near the end points or the solution experiences the global phenomenon of rapid oscillation throughout the entire interval. Typically, these problems arise very frequently in fluid dynamics, elasticity, quantum mechanics, chemical reactor theory and many other allied areas. In recent years, there are a wide class of special purpose methods available for solving the above type problems. But this field will be one of our future research works.

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