Set point stabilization of a 2DOF underactuated manipulator

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Abstract— **Controlling an underactuated manipulator with less actuators than degrees of freedom is a challenging problem, specifically when it is to force the underactuated manipulator to track a given trajectory or to be configurated at a specific position in the work space. This paper presents two controllers for the set point regulation of 2-DOF underactuated manipulators. The first one is a cascade sliding mode tracking controller while the second one uses an input-output feedback linearization approach. The first algorithm builds on an observation that an underactuated manipulator can be treated as two subsystems. Consequently, a cascade sliding mode tracking controller has been designed. Firstly, a sliding mode surface is designed for both subsystems, these two sliding surfaces represent a first layer in the design architecture. A second layer sliding mode surface is then constructed based on the first layer sliding surface. The cascaded sliding mode controller is therefore deduced in terms of Lyapunov stability theorem. Robustness issues to bounded disturbances are then investigated. In a second stage of the paper, the input output feedback linearization (IOFL) control is presented. The latter, is then mixed to the sliding mode control scheme for robustness issues.**
Simulation results on 2-DOF whirling pendulum are presented to demonstrate the effectiveness of the proposed approach.

Index Terms— **robust stabilization, sliding mode, underactuated manipulator, whirling pendulum, Input output feedback linearization.**

I. Introduction

Underactuated mechanical systems are characterized by the fact that they have less actuators than degrees of freedom. In these classes we find : the acrobat [1], inverted pendulum [2], pendubot [3], ball and beam system [4], aircraft, underwater, robot devises and legged humanoids with some passive joints. The underactuation arises by deliberate design for the purpose of reducing weight of the manipulator or might be caused by actuator failure. For fully actuated mechanical system, broad ranges of control techniques are used to improve their performance (adaptive, robust, optimal, etc). These techniques are possible because fully actuated systems possess a number of strong properties that facilitate the control design such as feedback linearization, passivity. The difficulty in controlling underactuated mechanisms is due to the fact that, the techniques developed for fully actuated systems cannot be used directly because the dimension of the input spatial , it makes difficult to carry out some controlling tasks so these system are not feedback linearizable [1], [5], and are not locally controllable at their equilibrium point [6]. Underactuated manipulators are manipulators with a number of passive joints, they belong to the class of second order nonholonomic systems which do not satisfy the Brockett's condition for smooth feedback stabilization [7]. The absence of actuators for some of the joints introduce non-holonomic constraints in the system. Control of passive manipulators have considerable challenges due to their higher nonlinear and coupled structures. Olfati [8] classified underactuated systems to eights classes and provided analytical tools that allow to deal with their control problems. In [9], Underactuated systems are classified into three types according to their control flow diagram representation, namely, the chain, tree and isolated-point structures. Despite the efforts of researchers in the last decade. A global controller to solve for control problems of all underactuated systems does not exist. Fantoni [10] solved the control of classes of underactuated mechanical systems based on energy approach and the passivity properties of the system. Park [11] divided the system on subsystems, applied a standard Sliding Mode Control and considered the zero dynamics for each closed loop subsystem. They introduced a semi globally stable control law defined by two layer sliding planes that verify the Lyapunov stability and applied it to swing up a pendubot. Qian and Yi [12] developed a sliding mode controller for a pendubot with a limited control torque and assumed that all state variables are equivalent to each other in a neighborhood of the control objective if parameters of the sliding mode surface are bounded. Zhuang [13] proposed a passivity-based variable structure controller but they used two types of control. The passive joints are controlled first then the active joint in second. Lin [14] proposed hierarchical fuzzy sliding mode controller to achieve decoupling control for underactuated systems. They defined individual sliding surfaces for each subsystem and coupled them using optimal coupling factor which is determined by fuzzy systems with optimization.

In this paper an Output Feedback Sliding Mode Control is designed to control Whirling pendulum. The pendulum is considered as two subsystems for each one we construct an attractive sliding mode surface. The objective is to construct a controller that tracks the desired point. The sliding

doi:10.4304/jcp.6.2.368-376

mode controller is applied to a new input of the system resulting from an input-output feedback linearization.

Our main contributions throughout this paper include: (i) a hierarchical sliding mode controller is proposed for the robust stabilization of two link underactuated robotic manipulator the whirling pendulum with friction. For both actuated and unactuated variables, different sliding surfaces are proposed.(ii) Demonstrating that the proposed controller guarantees not only the convergence of each sliding surfaces, but also of the whole system is stable and (iii) a hybrid controller based on sliding mode control and input output feedback linearization is proposed and implemented to whirling pendulum.

In section II, we'll present the dynamic of underactuated system and the dynamic of whirling pendulum. In section III, we'll give an overview about variable structure control and we'll present our control design. In section IV-A, we'll show the control design of input output feedback linearization and we present a new controller based on sliding mode control and input output stabilisation. In section V, we show the simulation results. In section VI, we'll conclude and discuss future works.

II. Dynamical Model

1) General dynamic form of Underactuated Manipulators: Consider the class of Underactuated mechanical for which the motion equations of can be expressed as :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_j}}\right) - \left(\frac{\partial L}{\partial x_j}\right) = Q_j^n \qquad j = 1, 2...n \quad (1)$$

where $x_j \in R$ is the position of the j- th generalized coordinates of the system, $\dot{x_j} \in R$ is the speed of the j-th coordinate of the system and Q_j^n is the vector forces acting on the generalized j-th coordinate. The dynamic model of a mechanical system (1) can be written as:

$$D(x)\ddot{x} + C(x, \dot{x})\dot{x} + F(\dot{x}) + G(x) = B(x)\tau \quad (2)$$

where $D(x)$ is the inertia matrix, $C(x, \dot{x})$ is the Coriolis and the centrifugal acceleration matrix and $G(x)$ represents the gravity terms, $B(x)$ is a matrix mapping the extern forces, $\tau \in R^m$ (m: number of actuated variable on the system) is the vector of applied torques, $F(\dot{x})$ is a friction force. Friction is a major problem faced by engineers when analyzing systems and predicting performances. Due to its highly nonlinear nature, frictions are difficult to incorporate in models when they are being simulated. They cause jitter, large tracking errors and limit cycling in moving bodies. After partitioning the inertia matrix accordingly, the dynamics can be cast in the state space representation as follows:

$$\ddot{x} = \alpha(x, \dot{x}) + D^{-1}(x)B(x)\tau$$

where $\alpha(x, \dot{x}) = -D^{-1}(x)(C(x, \dot{x})\dot{x} + F(\dot{x}) + G(x))$.

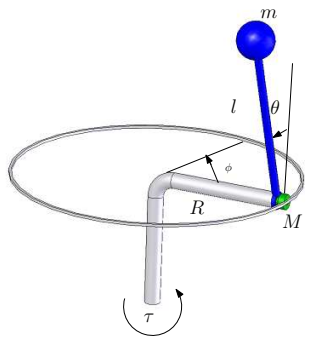

Figure 1. The Whirling pendulum.

2) The whirling pendulum: It consists of vertical shaft and a pendulum where the shaft is actuated by a motor, as shown in Figure 1 with the objective of balancing the pendulum in the inverted position whose suspension point is attached to the plane of the pendulum which is orthogonal to the radial arm of length R. We will take into account the frictional effects .
System parameters are presented as follows: l, I, m, M, R, τ, g: represent respectively the pendulum length, Bob inertia around its center of gravity, pendulum bob mass, whirling mass, radius of arm , shaft torque, gravitational acceleration. The position variables of the system are:
θ : angle of pendulum from the upward vertical.
ϕ : angle of mass from a fixed vertical plane.

Differential equations describing the dynamical behavior of the system can be derived using Lagrange formulation and are found to be [15]:

$$\begin{aligned} [MR^2 + m(R^2 + l^2\sin^2(\theta)]\ddot{\phi} + mRlcos\theta\ddot{\theta} & \\ +ml^2\sin(2\theta)\dot{\phi}\dot{\theta} - mlR\sin\theta\dot{\theta}^2 + F_{r0}\dot{\phi} + F_{fr} &= \tau_1 \\ mRlcos\theta\ddot{\phi} + (I + ml^2)\ddot{\theta} - ml^2\sin\theta cos\theta\dot{\phi}^2 & \\ -mglsin\theta + F_{r1}\dot{\theta} &= 0 \end{aligned}$$

Denoting:
$X = [x_1, x_2, x_3, x_4]^T$ where

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \phi \\ \dot{\phi} \\ \theta \\ \dot{\theta} \end{pmatrix}$$

then the matrix components of (2) are easily derived as:

$$D = \begin{pmatrix} F_{r0} + MR^2 + m(R^2 + l^2\sin^2\theta) & mRl\cos\theta \\ mRl\cos\theta & (I + ml^2) \end{pmatrix}$$

$$C = \begin{pmatrix} ml^2\dot{\theta}\sin(2\theta) & -mlR\dot{\theta}\sin\theta \\ -ml^2\dot{\phi}\sin\theta cos\theta & F_{r1} \end{pmatrix}$$

$$G = \begin{pmatrix} F_{fr} \\ -mgl\sin\theta \end{pmatrix}$$

$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The generalized mass matrix D is symmetric and positive definite for all X.

$F_{r0}\dot{\phi}$ is the friction force. It takes into account viscous frictions in the motor axis and in the pendulum joint $F_{r1}\dot{\theta}$. For the rotation of the arm, dry friction is included in the form of a simplified symmetric coulomb friction model:

$$F_{fr} = K_f(1 - \frac{2}{(exp^{(2k\dot{\phi})} + 1)})$$

The parameters F_{r0},F_{r1}, K_f and k was chosen as positif terms.

III. Design of Variable Structure Control

A. Variable Structure Control

The VSS design is based on the definition of a sliding mode which requires knowledge on the bounds of the system disturbances and uncertainties. Control laws in sliding mode are composed of two parts: the equivalent control law u_eq to drive the system to desired sliding surface is a "reaching" phase and a switching control u_sw to guarantee the system to reside on the sliding surface. The sliding mode control law is then written as

$$u = u_{eq} + u_{sw}$$

The control law is then designed such that the system state trajectories are driven to the sliding surfaces $S = 0$. Define the positive definite Lyapunov function as $V = \frac{1}{2}S^TS$. If its derivative with respect to time is negative semi-definite, the system is guaranteed to be stable, i.e

$$\dot{V} = S\dot{S} \leq 0 \tag{3}$$

where S is defined as

$$S = cx + \dot{x} \qquad c > 0 \tag{4}$$

$$e = x - x_d \ and \qquad \dot{e} = \dot{x} - \dot{x_d} \tag{5}$$

B. Control design

We consider the design of a sliding mode controller for the following second-order system:

$$\ddot{x} = f(x, \dot{x}, t) + b(x)u \qquad b \neq 0 \tag{6}$$

With the nonlinear-coupled expression (6) can be transformed into the following form:

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= f_1 + b_1(x)u \\ \dot{x_3} &= x_4 \\ \dot{x_4} &= f_2 + b_2(x)u \end{aligned} \tag{7}$$

$X = [x_1, x_2, x_3, x_4]^T$ is the state variable vector, u is the control inputs. $f_1(x), f_2(x)$ and $b_1(x), b_2(x)$ are nominal nonlinear functions. We can transform the system into two subsystems with state variable groups $(x_1, x_2), (x_3, x_4)$ for whose we construct the following linear functions as sliding surfaces, which we call first level sliding mode and are given as:

$$\begin{aligned} s_1 &= c_1x_1 + x_2 \\ s_2 &= c_2x_3 + x_4 \end{aligned} \tag{8}$$

c_1 and c_2 are real positive design parameters. Differentiating (8) and equalizing to zero we obtain the equivalents control laws as

$$\begin{aligned} u_{eq1} &= -\frac{c_1\dot{x_1} + f_1(x)}{b_1} \\ u_{eq2} &= -\frac{c_2\dot{x_3} + f_2(x)}{b_2} \end{aligned} \tag{9}$$

For the whole system (7) we define the second-level sliding surface as :

$$S = \alpha_1 s_1 + \alpha_2 s_2 \tag{10}$$

where α_1,α_2 are the sliding control parameters

$$u = u_{eq} + u_{sw} \tag{11}$$

where the equivalent control is selected as the sum of the two controls in (9) such that

$$u_{eq} = \frac{\alpha_1 b_1(x)u_{eq1} + \alpha_2 b_2(x)u_{eq2}}{\alpha_1 b_1(x) + \alpha_2 b_2(x)} \tag{12}$$

In the following we use the Lyapunov stability theory, to find the switching control law. The convergence of the sliding mode requires that

$$\dot{V} = S\dot{S} \leq -\eta sgn(S) - kS \leq 0 \tag{13}$$

$$\begin{aligned} \dot{V} &= S\dot{S} = S(\alpha_1\dot{s_1} + \alpha_2\dot{s_2}) \\ &= S(\alpha_1(c_1\dot{e_1} + \dot{e_2}) + \alpha_2(c_2\dot{e_3} + \dot{e_4})) \\ &= S(\alpha_1(c_1\dot{e_1} + f_1(x) + b_1(x)u) \\ &\quad +\alpha_2(c_2\dot{e_3} + f_2(x) + b_2(x)u)) \end{aligned}$$

Substituting (11), (12) in (13) gives:

$$\begin{aligned} \dot{V} &= S(\alpha_1(c_1\dot{e_1} + f_1(x) + b_1(x)(u_{eq} + u_{sw})) \\ &\quad +\alpha_2(c_2\dot{e_3} + f_2(x) + b_2(x)(u_{eq} + u_{sw})) \\ &= S(u_{sw}(\alpha_1 b_1(x) + \alpha_2 b_2(x))) \end{aligned} \tag{14}$$

then:

$$u_{sw}(\alpha_1 b_1(x) + \alpha_2 b_2(x)) = -\eta sgn(S) - kS$$

so

$$u_{sw} = \frac{-\eta sgn(S) - kS}{\alpha_1 b_1(x) + \alpha_2 b_2(x)}$$

The control law (11) becomes

$$u = \frac{\alpha_1 b_1(x)u_{eq1} + \alpha_2 b_2(x)u_{eq2}}{\alpha_1 b_1(x) + \alpha_2 b_2(x)} + \frac{-\eta sgn(S) - kS}{(\alpha_1 b_1(x) + \alpha_2 b_2(x))} \tag{15}$$

C. Stability and convergence

1) Stability of S : From (B) we the following sliding surfaces

$$s_1 = c_1 x_1 + x_2 \quad (16)$$
$$s_2 = c_2 x_3 + x_4 \quad (17)$$
$$S = \alpha_1 s_1 + \alpha_2 s_2 \quad (18)$$

whereα_1,α_2 are the sliding control parameters. The Lyapunov energy function can be define as follows

$$V(t) = \frac{1}{2} S^2$$
$$\dot{V} = S\dot{S} \leq -\eta \mid S \mid -kS^2 \leq 0 \quad (19)$$

Integrating both sides, we obtain

$$\int_0^t \dot{V} d\tau = \int_0^t (S\dot{S}) d\tau = \int_0^t (-\eta|S| - kS^2) d\tau$$

then

$$V(t) - V(0) = \int_0^t (-\eta|S| - kS^2) d\tau$$

$$V(t) = \frac{1}{2} S^2 = V(0) - \int_0^t (\eta|S| + kS^2) d\tau \quad (20)$$
$$\leq \quad V(0) < \infty$$

we can obtain that $S \in L_\infty$then

$$\sup_{t \geq 0} |S| = \| S \|_\infty < \infty$$

$$V(0) = V(t) + \int_0^t (\eta|S| + kS^2) d\tau \geq \int_0^t \eta(|S| + kS^2) d\tau$$

we can find that

$$\lim_{t \to \infty} \int_0^t (\eta|S| + kS^2) d\tau \leq V(0) < \infty \quad (21)$$

According to Barbalat's lemma when $t \to \infty$, $\eta|S| + kS^2 \to 0$ which means

$$\lim_{t \to \infty} S = 0$$

Then, the second layer sliding surface is asymptotically stable.

2) Stability of s_1 and s_2:

- from [16], (19) and (20) we have

$$\int_0^\infty S^2 d\tau = \int_0^\infty (\alpha_1 s_1 + \alpha_2 s_2)^2 d\tau$$
$$= \int_0^\infty (\alpha_1^2 s_1^2 + 2\alpha_1 \alpha_2 s_1 s_2 + \alpha_2^2 s_2^2) d\tau < \infty \quad (22)$$

we obtain (22) if:

$$sgn(\alpha_1 s_1) = sgn(\alpha_2 s_2) if s_1 s_2 \neq 0 \quad (23)$$

If (23) is satisfied we can get

$$\int_0^\infty s_i^2 d\tau < 0$$
$$s_i \in L_2 \quad (24)$$

- From (21) we obtain $S \in L_\infty$ and if (23) is satisfied we have

$$s_i \in L_\infty \quad (25)$$

- From (20) we have $S \in L_2$. Differentiating (17) ,(18):

$$\dot{s}_1 = c_1 \dot{x}_1 + \dot{x}_2$$
$$\dot{S} = \alpha_1 \dot{s}_1 + \alpha_2 \dot{s}_2$$

Then $\dot{x}_1, \dot{x}_2$ are respectively the velocity and acceleration are bounded and c_1 is defined as constant than:

$$\dot{s}_1 \in L_\infty$$

Let us prove that $\dot{s}_2 \in L_\infty$ by contradiction.
we have $\dot{S} = \alpha_1 \dot{s}_1 + \alpha_2 \dot{s}_2$ and if $\dot{s}_1 \notin L_\infty$ then $\dot{S} \notin L_\infty$ this is a contradicts because $\dot{S} \in L_\infty$. Thus, we can obtain

$$\dot{s}_i \in L_\infty \quad (26)$$

Form (24),(25) and (26), and according to Barbalat's lemma we have:

$$\lim_{t \to \infty} s_1 = 0 \quad (27)$$
$$\lim_{t \to \infty} s_2 = 0 \quad (28)$$

We conclude that the first level sliding surfaces converge asymptotically to zero.

D. Convergence of system states

Substitute (16) in (27) and (17) in (28) we have

$$\lim_{t \to \infty} c_1 x_1 + x_2 = 0; c_1 > 0 \quad (29)$$
$$\lim_{t \to \infty} c_2 x_3 + x_4 = 0; c_2 > 0 \quad (30)$$

the following two cases are possible:

- If $sign(x_1) = sign(x_2)$

$$c_1 x_1 + x_2 = 0 \Longleftrightarrow c_1 x_1 = x_2$$
$$(29) \Longleftrightarrow \lim_{t \to \infty} x_1 = \lim_{t \to \infty} x_2 = 0$$

and if $sign(x_3) = sign(x_4)$

$$(30) \Longleftrightarrow x_3 = x_4 \Longleftrightarrow lim_{t \to \infty} x_3 = lim_{t \to \infty} x_4 = 0$$

- If $sign(x_1) \neq sign(x_2)$

$$c_1 x_1 + x_2 = 0 \Longleftrightarrow x_2 = -c_1 x_1$$
$$\dot{x_1} = -c_1 x_1$$
$$x_1(t) < |x_1(t_0)| e^{-c1(t-t_0)}$$

and for

$$x_2 = \dot{x_1}$$
$$x_2(t) < c_1 |x_1(t_0)| e^{-c_1(t-t_0)}$$

This proves that x_1, x_2 are locally exponentially stable with a rate of convergence equal to c_1 then

$$\lim_{t \to \infty} x_1 = \lim_{t \to \infty} x_2 = 0 \quad (31)$$

Similarly when $sign(x_3) \neq sign(x_4)$ we prove that x_3, x_4 are locally exponentially stable with a rate of convergence equal to c_2 Then

$$\lim_{t\to\infty} x_3 = \lim_{t\to\infty} x_4 = 0 \tag{32}$$

For the whirling pendulum we have $x_1 = \phi - \phi_d$, $x_2 = \dot{\phi} - \dot{\phi_d}$, $x_3 = \theta - \theta_d$, $x_4 = \dot{\theta} - \dot{\theta_d}$.

So (29) and (31) give we give :

$$\lim_{t\to\infty} \phi = \phi_d, \lim_{t\to\infty} \dot{\phi} = \dot{\phi_d}$$

and $(30), (32) gives$

$$\lim_{t\to\infty} \theta = \theta_d, \lim_{t\to\infty} \dot{\theta} = \dot{\theta_d}$$

we conclude that all states converge to the desired states.

E. Robustness issues

If system (7) system is subjected to external disturbances, it becomes as follows

$$\begin{aligned} \dot{x_2} &= f_1(x) + b_1(x)u + d_1(t,x) \\ \dot{x_4} &= f_2(x) + b_2(x)u + d_2(t,x) \end{aligned}$$

$d_1(t,x)$ and $d_2(t,x)$ are bounded perturbations.

$$\dot{S} = -\eta sgn(S) - kS \tag{33}$$

Consider $V = \frac{1}{2}S^t S$ as a candidate Lyapunov function for (18). The sliding mode dynamics (33) can be viewed as a perturbed system :

$$\dot{x} = f(t,x) + d(t,x) \tag{34}$$

Where $x \in R^n$ and $d(t,x) \in R^n$ is a non-vanishing perturbation term for the nominal system $\dot{x} = f(t,x)$. From [17], it is shown that, if S satisfies the following properties such that

$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2 \tag{35}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3\|x\|^2 \tag{36}$$

$$\left\| \frac{\partial V}{\partial x} f(t,x) \right\| \leq c_4\|x\| \tag{37}$$

c_1, c_2, c_3 and c_4 are positive constant, then x is uniformly exponentially convergent and consequently state $x(t)$ verifies

$$\begin{aligned} \|x(t)| &\leq \left[\frac{V(t,x)}{c_1}\right]^{\frac{1}{2}} \\ &\leq \sqrt{\frac{c_2}{c_1}}\|x(t_0)\|e^{\frac{-c_3}{2c_2}(t-t_0)} \end{aligned}$$

Suppose now the perturbation term satisfies

$$\|d(t,x)\| \leq \frac{r}{b}\sigma \tag{38}$$

For all $t \geq 0$, all $x \in D \,/\, D = x \in R^n | \|x\| < r$. Then, for all $\| x(t_0) \| < (\sqrt{\frac{c_1}{c_2}} r)$ the solution of the perturbed system $x(t)$ satisfies:

$$\begin{aligned} \|x(t)\| &< \kappa\|x(t_0)\|e^{-\gamma(t-t_0)}, \forall t_0 \leq t \leq t_1 \\ \|x(t)\| &\leq b \qquad , \forall t > t_1 \end{aligned}$$

For $t \geq t_1$,

$$\kappa = \sqrt{\frac{c_2}{c_1}}, \gamma = \frac{c_3}{2c_2}, b = \frac{c_2}{c_3}\sqrt{\frac{c_2}{c_1}}\sigma \tag{39}$$

Applying (35), (36), (37) to (33), the expression of the perturbation can be expressed by :

$$d(t,S) = -\eta sgn(S) \tag{40}$$

with $\dot{S} = -kS + d(t,S)$, the nominal system $\dot{S} = -kS$ has an exponentially stable equilibrium point at $S = 0$, since $k > 0$. The norm of perturbation (40) is assumed to be bounded such that:

$$\|d(t,S)\| \leq \eta \tag{41}$$

Differentiating the Lyapunov function we get

$$\begin{aligned} \dot{V} &= S(-kS - \eta sgn(S)) \\ \dot{V} &\leq -k\|S\|^2 - \eta\|S\| \end{aligned}$$

with

$$c_1 = c_2 = \frac{1}{2}, c_3 = k, c_4 = 1 \tag{42}$$
$$b = \frac{\sigma}{k}$$

Substituting (42) in (38)we get the condition

$$\|d(t,S)\| \leq kr$$

We can conclude that $S \leq b = \frac{\sigma}{k}$. Then S is attractive, continuous and uniformly bounded.

IV. Output Feedback Sliding Mode Control

A. Input output feedback linearization

The controller is based in the theory of feedback input-output linearization (IOFL) [17]. IOFL is the classical way to control system described by the state space representation as follows:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{43}$$

The objective of this controller is a set point stabilization of the whirling pendulum at desired point y_d. We consider θ as the system output.

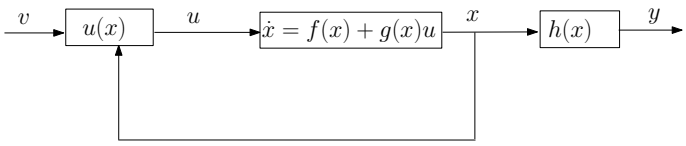

Figure 2. Input output feedback linearization

In order to obtain the input-output feedback linearization of (43) differentiating the output y with respect to

time yields [18]:

$$\begin{aligned} \dot{y} &= \frac{\partial h}{\partial x}\dot{x} \\ &= \frac{\partial h}{\partial x}[f(x)+g(x)u] \\ &= L_f h(x) + L_g h(x) u \\ \ddot{y} &= L_f^2 h(x) + L_g L_f h(x) u \end{aligned} \tag{44}$$

Applying a state feedback law that compensates the nonlinearities in the input-output behavior:

$$u = \frac{\upsilon - L_f^2 h(x)}{L_g L_f h(x)} \tag{45}$$

where υ is the new input. For the whirling pendulum we have:

$$\begin{aligned} L_f h(x) &= x_4 \\ L_f^2 h(x) &= f_2 \\ L_g h(x) &= 0 \\ L_g L_f h(x) &= b_2 \end{aligned}$$

Since $r = 2$ is the relative degree. For the control, we have

$$u = \frac{\upsilon - f_2}{b_2}$$
$$\theta \neq -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} ... \tag{46}$$

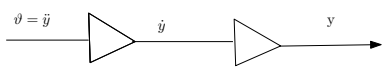

Figure 3. Input output feedback linearization

Then we have

$$\ddot{y} = \upsilon$$

The system is then globally linearized in its input-output behavior and a simple linear controller υ can be used to control the system output.

$$\upsilon = k_p(\theta_d - \theta) + k_d(\dot{\theta}_d - \dot{\theta}) \tag{47}$$

Because of the term of $\cos(\theta)$ in the denominator of equation (46), this control signal is defined in every position of the whirling except for the horizontal. The underactuated pendulum proves to be a system suitable for the application of (IOFL). The actuated DOF is globally stabilized using linear pole placement techniques.

B. Control design

In this part, we use a new controller based on sliding mode controller and input output feedback linearization. The objective of this controller is also a set point stabilization of the whirling pendulum at desired point y_d. The relative degree of system is equal to 2 , the control have the following form

$$u = \frac{\upsilon - f_2}{b_2} \tag{48}$$

and we have

$$\ddot{y} = \upsilon$$

Using the sliding mode control as a new control:

$$\begin{aligned} \upsilon &= u_{eq1} + u_{eq2} + \Delta_{eq} \\ \Delta_{eq} &= -K sign(S) \end{aligned} \tag{49}$$

where u_{eq1}, u_{eq2} are defined as (9) and S as (10).

C. Stability and convergence in the state space

The controller expressed in (49), we choose the Lyapunov function: $V = \frac{1}{2}S^T S$. Differentiating we get

$$\begin{aligned} \dot{V} &= S\dot{S} \leq 0 \\ &= S(\alpha_1(c_1\dot{x_1} + f_1(x) + b_1(x)\upsilon) \\ &+ \alpha_2(c_2\dot{x_3} + f_2(x) + b_2(x)\upsilon)) \end{aligned} \tag{50}$$

substitute (49) in (50)

$$\dot{V} = -K \parallel (S) \parallel \leq 0$$

Then the system has been stabilized by (49) and its system trajectory will approach the sliding surface on till converge to the origin.

The proof of the state convergence is similar to the the proof in the previous section.

V. Simulation Results

To demonstrate the effectiveness of the proposed approach, we consider the 2-DOF whirling pendulum in Figure 1.

System parameters are $M = 0.355$, $m = 0.25$ kg, $l = 26$ cm, $R = 52$ cm, $I = 0.53$.
The initial conditions of the whirling pendulum are $(\phi_0, \dot{\phi}_0) = (0.5, 0)$, $(\theta_0, \dot{\theta}_0) = (0, 0)$ and the expectations are $\phi_d = 0$, $\theta_d = 0$ and $\dot{\phi}_d = \dot{\theta}_d = 0$, $c_1 = 0.7$, $c_2 = 20$, $\alpha_1 = -3$, $\alpha_2 = 2$, $k = 25$, $\eta = 1$.

Figure 4-10 show respectively the system state responses when employing the SMC control. Figure 8 shows the response of the state when a constant disturbance is applied to the system input between t=13 sec and t=15 sec. It is clear that the proposed controller is able to reject a disturbances such an impulse. Figure 9 shows the response of the system when a sinusoidal-type disturbance $d(t) = 0.5.\sin(\pi/6.t)$, is applied to the system. Clearly

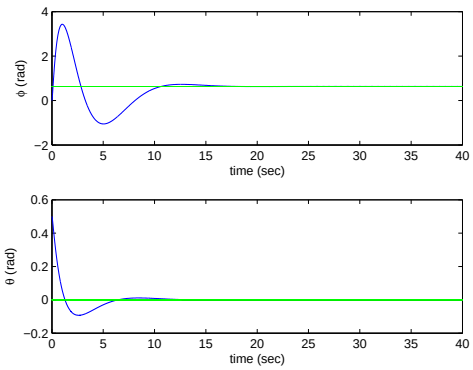

Figure 4. Time evolution of the pendulum angle without friction using (SMC)

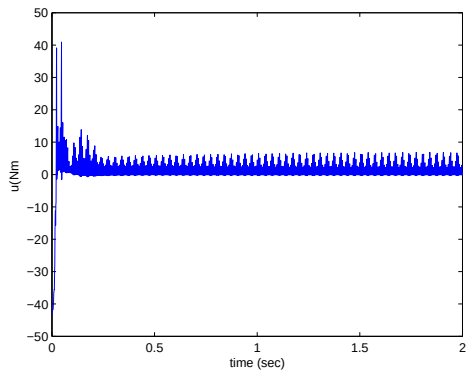

Figure 5. Control input time evolution using (SMC)

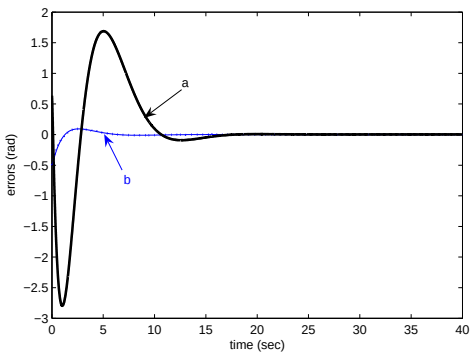

Figure 6. Time evolution of (a) errors of ϕ and (b) errors of θ using (SMC)

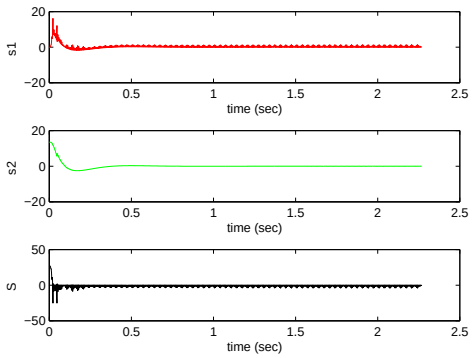

Figure 7. Time evolution of $s_1, s_2 and S$ respectively without friction (SMC)

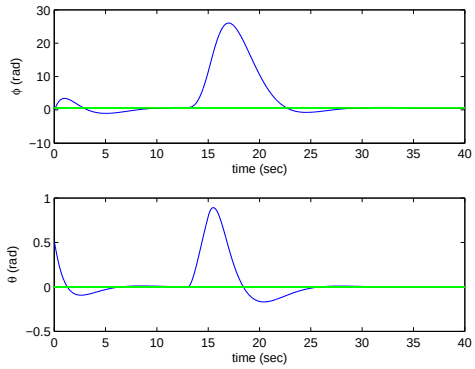

Figure 8. Time evolution of pendulum angle with impulse perturbation (SMC)

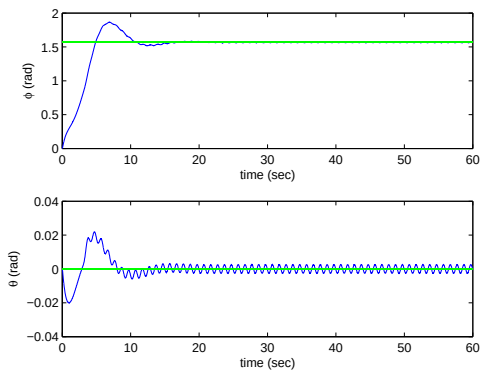

Figure 9. Time evolution of the pendulum angle with sinusoidal perturbation (SMC)

the system exhibits an acceptable behavior with respect to disturbances.

Figure 12 and 13 show the evolution of the pendulum states using the the hybrid controller. It'is obvious that the hybrid controller was able to overcome the effect of friction.

VI. Conclusion

In this paper, two algorithms for the set point stabilization of a 2-DOF underactuated manipulator are presented. Firstly a hierarchical VSS control for a class of second order underactuated manipulators has been developed. The control has been based on the construction of cascade sliding mode controller made up of two layer sliding surfaces. Based on Lyapunov analysis, this paper proved that the last sliding layer and consequently the system error tracking states converge asymptotically to zero.

Secondly a hybrid controller has been presented, combining the sliding mode control with input output feedback linearization control. The convergence of this controller has been proved.

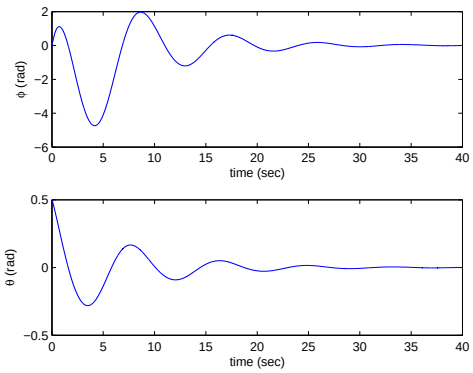

Figure 10. Time evolution of the pendulum angle with friction (SMC)

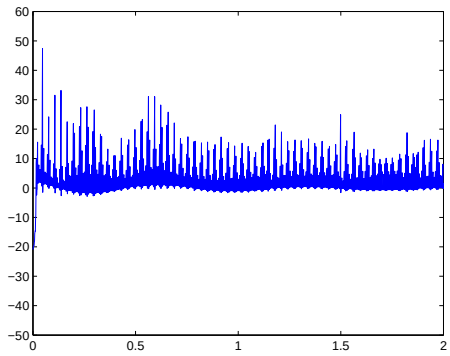

Figure 11. Control input time evolution with friction (SMC)

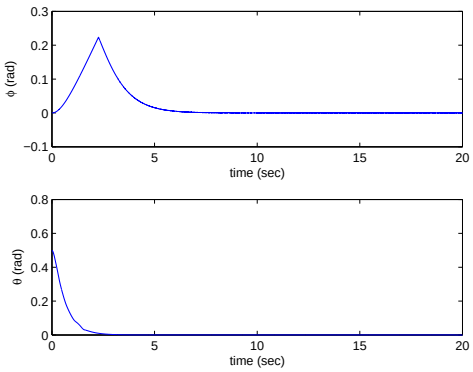

Figure 12. Time evolution of the pendulum angle without friction (IOFL)

Simulation results on a whirling pendulum are presented to demonstrate the effectiveness of the proposed controller. Future work under investigation include: an adaptive sliding mode controller for system with parametric variations and the design of a new sliding mode strategy.

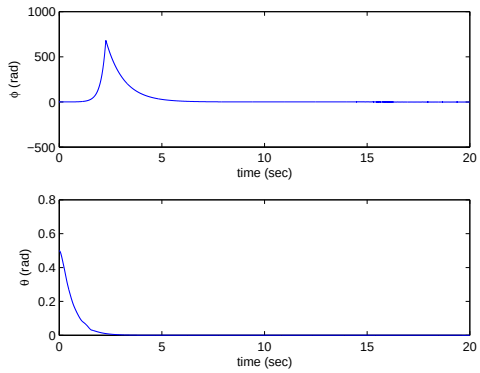

Figure 13. Time evolution of the pendulum angle with friction (IOFL)

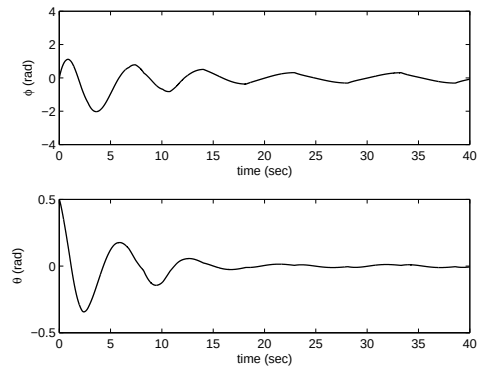

Figure 14. Time evolution of the pendulum angle with friction using the propoded hybrid controller

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