On Digital Filtering of Band-limited Signals Using Lower Sampling Rates

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***Abstract*—Filtering of multi-band bandlimited signals by means of a linear digital filter with one or more stopbands is explored. The main goal of the paper is to demonstrate that such a task can be accomplished using sampling rates lower than Landau rate, where the Landau rate is defined as the total bandwidth of the input signal. In order to reach such low rates Periodic Nonuniform Sampling is employed. We show that the proposed filtering method is most efficient when bandpass and multiband filtering is required. Necessary and sufficient conditions for filtering are derived, and an algorithm for designing PNS grids that allow sub-Landau sampling and filtering is proposed. Reconstruction systems are discussed and experimental examples are presented, which confirm the feasibility of the approach.**

***Index Terms*—digital filter, periodic nonuniform sampling, sampling rate reduction.**

I. Introduction

Classical DSP theory and applications are based on the Whittaker-Kotelnikov-Shannon theorem [1] which states that if a signal's spectrum is entirely placed inside the frequency interval $F = [-f_0, f_0)$, then the signal can be reconstructed from uniformly distributed samples taken at least at the rate $f_S = 2f_0$, known as the Nyquist rate. When lowpass signals are processed, the minimum sampling rate required for signal reconstruction is equal to the total bandwidth of the signal which is $B = 2f_0$. If lowpass filtering of such signal is needed a sub-Nyquist uniform sampling rate could be used if the resulting aliasing is present only in the rejected stopbands. However, if highpass filtering is required then the minimum uniform sampling rate is the Nyquist rate, or $B = 2f_0$.

In the case of bandpass and multiband signals, i.e. for which $B < 2f_0$, uniform sampling, reconstruction and filtering at the rate B is possible only if there is no aliasing within the original signal's spectrum $X(f)$. In that situation the minimum achievable sampling rate varies between Landau and Nyquist when bandpass sampling is deployed [2].

In order to reach lower than Nyquist sampling rates when processing bandpass and multiband signals, different, nonuniform sampling schemes were explored, amongst which was Periodic Nonuniform Sampling (PNS). Various authors ([3], [4], [5], and [6]) showed that PNS can be used for sampling and reconstruction of bandpass and multiband signals using lower than Nyquist rates. Landau [7] proved that the minimum sampling rate for arbitrary sampling and reconstruction of multiband signals asymptotically reaches the signal's total bandwidth B which may be significantly lower than the Nyquist rate. This result was exploited and several methods for sampling and reconstruction of passband and multiband signals using nonuniform sampling were introduced ([8], [9], [10], [11], and [12]).

All of the aforementioned work has focused primarily on the sampling and reconstruction of bandlimited signals. However, digital filtering using lower than Landau sampling rates has not been studied widely. This work explores a practical topic – digital filtering of band-limited signals by means of linear digital filter characterized by one or more stopbands. In this case the output signal's bandwidth will be lower than the input's bandwidth which implies that lower than Landau sampling rate could possibly be used. The employed sampling scheme is PNS. We show that many digital filtering problems could be effectively tackled with by using the proposed method, which is most efficient when digital filtering of bandpass and multiband signals is required.

The paper is organized as follows. Section II presents the analysis and definition of the proposed method. Section III outlays the necessary and sufficient conditions for perfect digital filtering. In section IV we derive a practical algorithm for designing optimal PNS sequences allowing for digital filtering using lower than Landau sampling rates. In section V several filter implementation schemes are compared. Section VI contains several numerical examples. Section VII presents the conclusions of the study.

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II. PNS AND THE PROPOSED METHOD

PNS is a sampling scheme with sampling instants taken at times $t_{l+qL} = t_l + qT$, where $l = 0,...,L\text{-}1$. This sampling sequence is periodic with period T, L denotes the number of sampling instants per period: $0 = t_0 < t_1 < ... < t_{L\text{-}1} < T$, and q is an arbitrary integer indicating to which period the given sample time belongs. Let us assume that each sampling time t_n, $n = 0, \pm 1, \pm 2 \pm 3, ...$ is a multiple of a short interval d:

$$t_n = c_n d, n \in \mathbf{I}. \tag{1}$$

A PNS scheme can be described as L parallel branches each using uniform sampling with period T but shifted in time with respect to each other. This is illustrated in Figure 1.

We begin our analysis by dividing the frequency axis into an infinite number of frequency cells F_k:

$$F_k = \left[\frac{k\text{-}0.5}{T}, \frac{k+0.5}{T}\right). \tag{2}$$

Each frequency cell has length $|F_k| = 1/T$. Here T is a long enough period such that its inverse $f_T = 1/T$ provides satisfactory resolution in the frequency domain. We assume that the signal occupies a finite number of not necessary adjacent cells $F_{n_m} : F_{n_1}, F_{n_2}, ..., F_{n_M}$. M is the number of the occupied cells. The numbers n_M do not have to be put in a monotonic order. Let the first M_P cells represent the bands of the processed signal which will be preserved. For simplicity, we assume that the frequency response of the filter at these bands is one. The remaining M_S frequency cells coincide with the stopbands of the filter; hence the filter's frequency response at these bands is zero. Here, $M = P + S$. We request that each cell F_{n_m} is fully occupied by either the passbands or the stopbands of the filter. (An example of the proposed analyzed scheme is illustrated in Figure 2.). The spectrum of the processed signal is thus:

$$X(f) = \sum_{m=1}^{M} X_{n_m}\left(f - \frac{n_m}{T}\right), \tag{3}$$

where the subspectra $X_{n_m}(f)$ are defined by:

$$X_{n_m}(f) = \begin{cases} X\left(f + \frac{n_m}{T}\right), f \in F_{n_m} \\ 0, \text{otherwise.} \end{cases}. \tag{4}$$

The discrete spectrum $X_{d,l}(f)$ in the $l-$ th branch of the PNS system is:

$$X_{d,l}(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(f + \frac{k}{T}\right) e^{j2\pi \frac{k}{T} t_l}. \tag{5}$$

Substituting (3) and (4) in (5) we find the relation between the components $X_{n_1}(f),...,X_{n_M}(f)$, and the discrete-time spectra $X_{d,0}(f),...,X_{d,L\text{-}1}(f)$. Then we confine our analysis to the baseband frequency interval $f \in [-0.5/T, 0.5/T)$:

$$X_{d,l}(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty}\sum_{m=1}^{M} X_{n_m}\left(f + \left(\frac{k - n_m}{T}\right)\right)\left(e^{j2\pi \frac{t_l}{T}}\right)^k. \tag{6}$$

Considering the fact that if $f \in [f_0, f_0 + 1/T)$, $X_{n_m}(f - k/T)$ takes nonzero values inside this interval only if $k = 0$, we can simplify (6):

$$X_{d,l}(f) = \frac{1}{T}\sum_{m=1}^{M} X_{n_m}(f) z_l^{n_m}. \tag{7}$$

where $z_l = e^{j2\pi \frac{t_l}{T}}$. Note that $z_0 = 1$. Expanding (7) to all L branches of the PNS scheme and presenting it in the matrix form, we can write:

$$\begin{bmatrix} X_{d,0}(f) \\ X_{d,1}(f) \\ ... \\ X_{d,L\text{-}1}(f) \end{bmatrix} = \frac{1}{T}\underbrace{\begin{bmatrix} 1 & ... & 1 & | & 1 & ... & 1 \\ z_1^{n_1} & ... & z_1^{n_P} & | & z_1^{n_{P+1}} & ... & z_1^{n_M} \\ ... & & ... & | & ... & & ... \\ z_{L\text{-}1}^{n_1} & ... & z_{L\text{-}1}^{n_P} & | & z_{L\text{-}1}^{n_{P+1}} & ... & z_{L\text{-}1}^{n_M} \end{bmatrix}}_{\mathbf{Z}_P \quad \mathbf{Z}_S} \begin{bmatrix} \mathbf{X}_P \\ --- \\ \mathbf{X}_S \end{bmatrix}, \tag{8}$$

or in short:

$$\mathbf{X}_d = \frac{1}{T}\left(\mathbf{Z}_P \mathbf{X}_P + \mathbf{Z}_S \mathbf{X}_S\right), \tag{9}$$

where

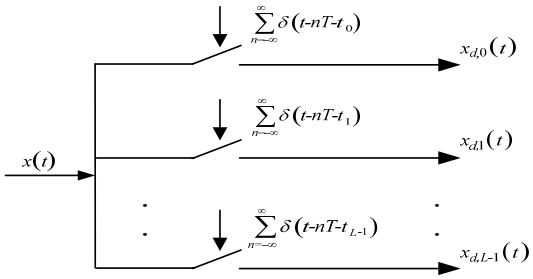

Figure 1. Periodic Nonuniform Sampling scheme.

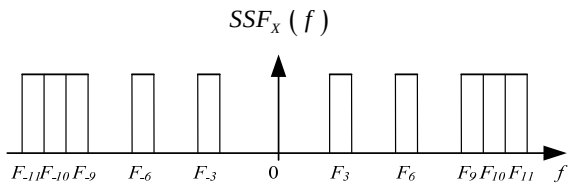

Figure 2. Frequency domain division of input signal's spectrum.

$$\mathbf{X}_P = \begin{bmatrix} X_{n_1}(f) \\ \dots \\ X_{n_P}(f) \end{bmatrix}, \ \mathbf{X}_S = \begin{bmatrix} X_{n_{P+1}}(f) \\ \dots \\ X_{n_M}(f) \end{bmatrix},$$

$$\mathbf{Z}_P = \begin{bmatrix} 1 & \dots & 1 \\ z_1^{n_1} & \dots & z_1^{n_P} \\ \dots & \dots & \dots \\ z_{L-1}^{n_1} & \dots & z_{L-1}^{n_P} \end{bmatrix}, \text{ and } \mathbf{Z}_S = \begin{bmatrix} 1 & \dots & 1 \\ z_1^{n_{P+1}} & \dots & z_1^{n_M} \\ \dots & \dots & \dots \\ z_{L-1}^{n_{P+1}} & \dots & z_{L-1}^{n_M} \end{bmatrix}.$$

The solution to (9) with respect to $\mathbf{X}_P$ will obtain the spectral components $X_{n_1}(f),\dots,X_{n_M}(f)$, thus allowing for reconstruction of the passband frequencies. However, the main problems we address here are:

1) Knowing the vector $\mathbf{X}_d$, and the matrices $\mathbf{Z}_P$ and $\mathbf{Z}_S$, under what conditions can we find $\mathbf{X}_P$?

2) What is the smallest number L of sampling instants inside one period T for which all solutions to (9) give correct results for $\mathbf{X}_P$?

III. Conditions for Perfect Filtering

The following theorem summarizes the necessary and sufficient conditions required for the recovery of $\mathbf{X}_P$.

Theorem: If a band-limited signal *x(t)*, described in the context of (3) and (4), is sampled using Periodic Nonuniform Sampling scheme, the necessary and sufficient conditions for its perfect filtering with respect to $\mathbf{X}_P$ are:

$$\text{rank}\left[\mathbf{X}_d, \mathbf{Z}_P, \mathbf{Z}_S\right] = \text{rank}\left[\mathbf{Z}_P, \mathbf{Z}_S\right],$$
$$\text{rank}\left[\mathbf{Z}_P\right] = P,$$
$$\text{rank}\left[\mathbf{Z}_P | \mathbf{Z}_S\right] = \text{rank}\left[\mathbf{Z}_P\right] + \text{rank}\left[\mathbf{Z}_S\right].$$

Proof of Theorem: We remark that because of the structure of the vector $\mathbf{X}_d$ and the matrices $\mathbf{Z}_P$ and $\mathbf{Z}_S$, the equation

$$\mathbf{X}_d = \frac{1}{T}\left(\mathbf{Z}_P\hat{\mathbf{X}}_P + \mathbf{Z}_S\hat{\mathbf{X}}_S\right), \tag{10}$$

can always be solved with respect to $\hat{\mathbf{X}}_P$ and $\hat{\mathbf{X}}_S$. We only need to choose $\hat{\mathbf{X}}_P = \mathbf{X}_P$ and $\hat{\mathbf{X}}_S = \mathbf{X}_S$. We therefore expect that:

$$\text{rank}\left[\mathbf{X}_d, \mathbf{Z}_P, \mathbf{Z}_S\right] = \text{rank}\left[\mathbf{Z}_P, \mathbf{Z}_S\right], \tag{11}$$

Next, the conditions under which all solutions to (10) are such that:

$$\hat{\mathbf{X}}_P = \mathbf{X}_P \tag{12}$$

have to be found. Then we will be able to reconstruct the filtered signal. Let us denote $\Delta\mathbf{X}_P = \hat{\mathbf{X}}_P - \mathbf{X}_P$, and $\Delta\mathbf{X}_S = \hat{\mathbf{X}}_S - \mathbf{X}_S$. We prove that all solutions to (10) satisfy (12) if and only if:

1) The only solution of $\mathbf{Z}_P\Delta\mathbf{X}_P = 0$ is $\Delta\mathbf{X}_P = 0$;

2) All solutions to $\mathbf{Z}_P\Delta\mathbf{X}_P = -\mathbf{Z}_S\Delta\mathbf{X}_S$ satisfy $\mathbf{Z}_P\Delta\mathbf{X}_P = -\mathbf{Z}_S\Delta\mathbf{X}_S = 0$.

Both conditions are necessary. If the first condition is not satisfied then we could find nonzero $\Delta\mathbf{X}_P$ such that $\mathbf{Z}_P\Delta\mathbf{X}_P = 0$ and construct the following solution (10): $\hat{\mathbf{X}}_P = \mathbf{X}_P + \Delta\mathbf{X}_P$, $\hat{\mathbf{X}}_S = \mathbf{X}_S$, which does not satisfy (12). Similarly, if the second condition is not satisfied, we could find a solution to $\mathbf{Z}_P\Delta\mathbf{X}_P = -\mathbf{Z}_S\Delta\mathbf{X}_S$ such that $\mathbf{Z}_P\Delta\mathbf{X}_P = -\mathbf{Z}_S\Delta\mathbf{X}_S \neq 0$. By choosing $\hat{\mathbf{X}}_P = \mathbf{X}_P + \Delta\mathbf{X}_P$ and $\hat{\mathbf{X}}_S = \mathbf{X}_S + \Delta\mathbf{X}_S$ we get a solution to (10) that does not comply with (12).

The above conditions are also sufficient. We demonstrate this by proving that if there exists a solution to (10) which contradicts (12), then at least one of the above conditions is violated. Let $\mathbf{X}_d = \mathbf{Z}_P\hat{\mathbf{X}}_P + \mathbf{Z}_S\hat{\mathbf{X}}_S$, and $\hat{\mathbf{X}}_P \neq \mathbf{X}_P$. Subtracting (12) from (10) gives $\mathbf{Z}_P\left(\mathbf{X}_P - \hat{\mathbf{X}}_P\right) = -\mathbf{Z}_S\left(\mathbf{X}_S - \hat{\mathbf{X}}_S\right)$. If the second condition is to be satisfied, then $\mathbf{Z}_P\left(\mathbf{X}_P - \hat{\mathbf{X}}_P\right) = 0$. Consequently, if the first condition is satisfied, we get $\hat{\mathbf{X}}_P = \mathbf{X}_P$, which contradicts our earlier assumption that $\hat{\mathbf{X}}_P \neq \mathbf{X}_P$. We note that it is necessary that the matrix $\mathbf{Z}_P$ has full column rank. If this was not satisfied then there would be nonzero values $\Delta\mathbf{X}_P$ such that $\mathbf{Z}_P\Delta\mathbf{X}_P = 0$. In such cases, if $\mathbf{X}_P$ and $\mathbf{X}_S$ solve (9) then $\mathbf{X}_d = \mathbf{Z}_P\left(\hat{\mathbf{X}}_P - \Delta\mathbf{X}_P\right) + \mathbf{Z}_S\mathbf{X}_S$. Having $\mathbf{Z}_P\Delta\mathbf{X}_P = 0$, we get $\mathbf{X}_d = \mathbf{Z}_P\hat{\mathbf{X}}_P + \mathbf{Z}_S\mathbf{X}_S$, which contradicts (9) and (10). □

Resorting on the Theorem, we can find $\mathbf{X}_P$ by:

$$\mathbf{X}_P = T\mathbf{\Phi}_P\mathbf{X}_d, \tag{13}$$

where:

$$\mathbf{\Phi}_P = \mathbf{Z}_P^{\mathrm{T}}\left(\mathbf{Z}_P\mathbf{Z}_P^{\mathrm{T}} + \mathbf{Z}_S\mathbf{Z}_S^{\mathrm{T}}\right)^{-1}. \tag{14}$$

Taking the inverse Fourier transform of $\mathbf{X}_P$, we derive:

$$\mathbf{X}_P = T\mathbf{\Phi}_P\mathbf{X}_d. \tag{15}$$

where $\mathbf{x}_P = [x_{n_1}(t),\dots,x_{n_P}(t)]'$ contains the samples carrying information of each passband component, and $\mathbf{x}_d = [x_{d,0}(t),\dots,x_{d,L-1}(t)]'$.

IV. Design of Optimal PNS Sequences

In this section we propose an algorithm that aims at reducing the values of L for which all solutions to (9) still give correct results for $\mathbf{X}_P$.

The third condition of the Theorem states that $\text{rank}\left[\mathbf{Z}_P \mid \mathbf{Z}_S\right] = \text{rank}\left[\mathbf{Z}_P\right] + \text{rank}\left[\mathbf{Z}_S\right]$. Hence:

$$P+1 \le L \le P + \text{rank}\left[\mathbf{Z}_S\right]. \tag{16}$$

To reduce the value of L, we have to find sampling instants t_l such that $\text{rank}\left[\mathbf{Z}_S\right]$ is minimum, and $\text{rank}\left[\mathbf{Z}_P\right] = P$. Or, in other words, find sampling instants t_l such that $\mathbf{Z}_S$ has as many pairs of linearly dependent columns as possible, while all columns of $\mathbf{Z}_S$ are linearly independent from the columns of $\mathbf{Z}_P$. $\mathbf{Z}_S$ can be presented as:

$$\mathbf{Z}_S = \begin{bmatrix} 1 & \dots & 1 \\ e^{j2\pi t_1 n_{p+1}} & \dots & e^{j2\pi t_1 n_M} \\ \dots & & \dots \\ e^{j2\pi t_{L-1} n_{p+1}} & \dots & e^{j2\pi t_{L-1} n_M} \end{bmatrix}. \tag{17}$$

Since the first element of each column of $\mathbf{Z}_S$ is one, then the pairs of linearly dependent columns of $\mathbf{Z}_S$ have to be identical. Comparing the first two columns of $\mathbf{Z}_S$, we have:

$$\begin{aligned} t_1 n_{P+1} &= t_1 n_{P+2} + k_1 \\ &\cdot \\ &\cdot \\ t_{L-1} n_{P+1} &= t_{L-1} n_{P+2} + k_{L-1} \end{aligned} \tag{18}$$

Denoting

$$\Delta_{p+1,p+2} = n_{p+1} - n_{p+2}, \tag{19}$$

we write

$$\Delta_{p+1,p+2} t_l \in \mathbf{I},\ l = 0,\dots,L\text{-}1. \tag{20}$$

Similarly, $\Delta_{p+2,p+3} t_l \in \mathbf{I}$, $\Delta_{p+3,p+4} t_l \in \mathbf{I}$, etc. Having as many $\Delta_{i,j} t_l \in \mathbf{I}$, $i,j = p+1,\dots,M$ as possible will minimize $\text{rank}\left[\mathbf{Z}_S\right]$, and hence L. Referring back to (1), we have $\Delta_{i,j} t_l = \Delta_{i,j} c_l d \in \mathbf{I}$, $c_l \in \mathbf{I}$. There is no known analytical expression for finding the optimum d which will guarantee finding the minimum L, and thus reaching the minimum average sampling rate $f_{S_{ave}} = L/T$. However, an exhaustive search for d would be confined to a finite number of iterations since t_l and T are multiples of d.

Based on the conclusions drawn above, we propose the following algorithm for optimal PNS sequence design:

Step 1: Choice of period T: T must be a long enough period which will allow for dividing the input and output signals' spectra in the context of (3) and (4). A good choice for an initial T can be the least common multiple of the spectral support functions' frequencies.

Step 2: Choice of d and L: choose d such that $d = T/N, N \in \mathbf{I}$. A practical guide to finding an optimal d is to check if after uniform sampling of the signal with period d the passbands should be alias-free. The aim is to create a "constructive aliasing" which concentrates in the rejected bands, or the stopbands of the filter. The total bandwidth of all spectral components inside the interval $f \in \left[-0.5/d, 0.5/d\right)$ will present the minimum number of sampling instants L inside one period T, and hence the minimum average sampling rate $f_{S_{ave}} = L/T$.

Step 3: Choice of t_l: choose t_l such that $t_l = c_l d,\ c_l \in \mathbf{I}$.

Step 4: Necessary and sufficient conditions check : check if the matrices $\mathbf{Z}_P$ and $\mathbf{Z}_S$, constructed with the chosen T, d and t_l, meet the conditions from the Theorem. If not, restart the algorithm from Step 2 by choosing a different d, or from Step 1 by choosing a different period T.

V. Implementation of the Filter

We propose and compare two filter implementation systems. The first realization is described as follows. After sampling the input signal using PNS with period T and sampling instants t_l, derived using the algorithm, we upsample the signals $x_{d,l}(t)$ to a sampling rate $f_d = 1/d = R/T$, where d is defined by (1) and found with the use of the outlined algorithm. The reconstruction matrix is calculated using (11) and each upsampled $x_{d,l}(t)$ is multiplied by the corresponding element of $\mathbf{\Phi}_P$. Subsequently, each $\mathbf{x}_P(t)$ is lowpass filtered in the baseband frequency interval $f \in \left[-0.5/T, 0.5/T\right)$, and frequency-shifted to the original band position. The reconstruction formula is thus:

$$x(t) = T \sum_{q=-\infty}^{\infty} \sum_{l=0}^{L\text{-}1} \sum_{p=1}^{P} \left[\mathbf{\Phi}_P\right]_{pl} x_{d,l}\left(t_l + qT\right) \phi_p\left(t - t_l - qT\right), \tag{21}$$

where

$$\phi_p(t) = e^{j2\pi \frac{n_p}{T} t} \frac{R}{T} \text{sinc}\left(\frac{t}{T}\right). \tag{21a}$$

We will denote this filter implementation scheme as Filter A which is shown in Figure 3. Note that when real-valued signals are processed, the Filter A system can be simplified to:

$$x(t) = x_r^+(t) + T \sum_{q=-\infty}^{\infty} \sum_{l=0}^{L-1} \left[\mathbf{\Phi}_P\right]_{0l} x_{d,l}\left(t_l + qT\right) \phi_0\left(t - t_l - qT\right), \tag{22}$$

where

$$x_r^+(t) = 2\mathrm{Re}\left\{ T \sum_{q=-\infty}^{\infty} \sum_{l=0}^{L-1} \sum_{p_+} \left[\mathbf{\Phi}_P\right]_{p_+ l} x_{d,l}\left(t_l + qT\right) \phi_{p_+}\left(t - t_l - qT\right) \right\}, \tag{22a}$$

$$\phi_{p_+}(t) = e^{j2\pi \frac{n_{p_+}}{T} t} \frac{R}{T} \mathrm{sinc}\left(\frac{t}{T}\right), \tag{22b}$$

and

$$\phi_0(t) = \frac{R}{T} \mathrm{sinc}\left(\frac{t}{T}\right). \tag{22c}$$

(22a) gives the positive frequency cells' components denoted by p_+ , whereas the second term in the sum of (22) is the baseband frequency cell component, if present and filtered. p_+ can be replaced by p_- with the same result applicable if the negative frequency cells are used for filtering. The reconstruction scheme will be denoted as Filter A1.

The second realization of the filter is a direct extension of the reconstruction method for PNS [11]. If the implementation is performed along the branches of PNS, that is there is a synthesis filter $\Psi_l(f)$ after each upsampler, the implementation scheme can be described as:

$$x(t) = T \sum_{q=-\infty}^{\infty} \sum_{l=0}^{L-1} \sum_{p=1}^{P} x_{d,l}\left(t_l + qT\right) \psi_l\left(t - t_l - qT\right), \tag{23}$$

where the filters $\psi_l(t)$ are described by:

$$\Psi_l(f) = \begin{cases} \frac{R}{T}\left[\mathbf{\Phi}_P\right]_{pl} e^{j2\pi \frac{t_l n_p}{T}}, & f \in F_{n_p} \\ 0, & \text{otherwise} \end{cases}, \tag{23a}$$

and $\Psi_l(f)$ is the Fourier transform of $\psi_l(t)$. The latter reconstruction scheme is shown in Figure 4, and will be denoted as Filter B. The three filter implementation schemes are equivalent because they produce identical results, as will be later confirmed by numerical examples. The difference is in the filters that are used. Filter A and Filter A1 schemes use one lowpass filter for each passband, whereas Filter B scheme uses L passband filters with piece-wise constant frequency response. In practice causal, linear-phase FIR or IIR filters could be used, which introduce some delay and distortion. Therefore, additional care must be taken that the group delays in each branch of the filter structure are equalized.

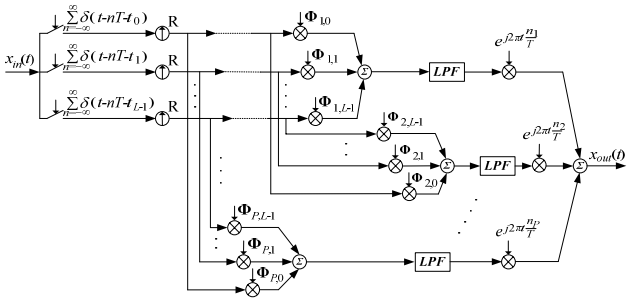

Figure 3. Digital filter implementation – Filter A.

VI. Numerical Examples

In this section we use several examples to show the feasibility of the method. We consider lowpass filtering of a baseband signal, and bandpass filtering of a multiband signals. In all of the examples we use normalized frequencies, and the simulations are performed over a periodogram taken over a window [0,105T).

Due to the time-varying nature of the proposed filtering approach, the examples compare the output of an ideal filter against the output of the explored PNS filter schemes. All filters have been designed and implemented in software environment, and in each example the filters are of similar orders and have similar transition bands, but different passbands, stopbands and attenuation specifications, respectively. This shows that the quality of filtering of the overall systems highly depends on the quality of the sub-filters.

Example 1 – Lowpass filtering of a baseband signal

Consider a test signal comprised of sinusoids placed inside the following spectral support:

$$SSF_{in} = \left(-5.5, 5.5\right) \text{ MHz}.$$

Let the filter's passband be:

$$SSF_{out} = \left(-2.5, 2.5\right) \text{ MHz}.$$

The Nyquist uniform sampling rate for reconstruction of this signal is $f_{Nyquist} = 2 f_{\max} = 2*5.5 = 11$ MHz. The Landau rate, or the minimum average sampling rate for sampling and reconstruction of the input signal, is equal to the Nyquist rate, or $f_{Landau} = 11$ MHz. However, such lowpass filtering of this signal can be performed at a lower uniform sampling rate $f_{opt} = \left(B_{in} + B_{out}\right)/2 = 8$ MHz, where B_{in} and B_{out} are the total bandwidths of the input and output signals, respectively. Initiating the algorithm described in section IV, we choose $T = 1$ μs. Hence, the input and output spectral supports can be represented as: $F_{in} = \left[-5 \;\; -4 \;\; -3 \;\; -2 \;\; -1 \;\; 0 \;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5\right]$, and $F_{out} = \left[-2 \;\; -1 \;\; 0 \;\; 1 \;\; 2\right]$. We choose $d = 1/f_{opt} = 1/8$ μs, as uniform sampling of the input signal with rate $f_d = 1/d = 8$ MHz does not induce aliasing in the passbands. Uniform sampling of the input signal at such rate shows that the total bandwidth of the

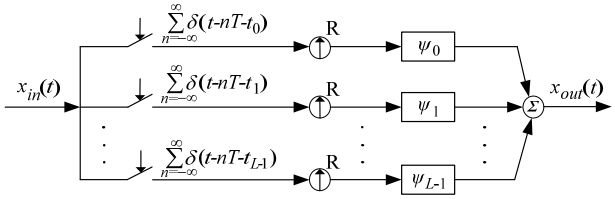

Figure 4. Digital filter implementation – Filter B.

spectral components in the interval $[-4,4)$ MHz is 8 MHz. Hence, we choose $L=8$. The vector t_l of the sampling instants inside a period T is then selected to be $t_l=[0\ \ 1d\ \ 2d\ \ 3d\ \ 4d\ \ 5d\ \ 6d\ \ 7d]$, as this is the only available choice for the sampling instants for the chosen T and d. Note that in this particular case the sampling sequence is uniform, and the filter structure is effectively a polyphase representation. Step 4 of the proposed algorithm confirms the proper choice of T, d, and t_l. In this case the proposed approach does not improve on the sampling rate, and filtering of the signal using uniform sampling would be preferable.

Example 2 – Bandpass filtering of a multiband signal

Consider a test signal of sum of sinusoids which has the following spectral support:

$$SSF_{in}=(-7.5,-5.5)\cup(-2.5,-1.5)\cup(-0.5,0.5)\cup$$
$$\cup(1.5,2.5)\cup(5.5,7.5)\text{ MHz.}$$

Let the filter passbands be :

$$SSF_{out}=(-6.5,-5.5)\cup(-2.5,-1.5)\cup(1.5,2.5)\cup(5.5,6.5)\text{ MHz.}$$

Figure 10 shows the power spectrum density of the input and output signals. The Nyquist uniform sampling rate for this signal is $f_{Nyquist}=2f_{\max}=2*7.5=15$ MHz. The Landau rate, or the minimum average sampling rate for sampling and reconstruction of the input signal, is $f_{Landau}=7$ MHz. Initiating the algorithm described in section IV, we choose $T=1$ μs. Hence, the input and output spectral supports can be represented as: $F_{in}=[-7\ \ -6\ \ -2\ \ 0\ \ 2\ \ 6\ \ 7]$, and $F_{out}=[-6\ \ -2\ \ 2\ \ 6]$. We choose $d=1/7$ μs, as uniform sampling of the input signal at rate $f_d=7$ MHz does not induce aliasing in the passbands. Also, uniform sampling of the input signal with such rate shows that the total bandwidth of the spectral components in the interval $[-7/2,7/2)$ MHz is 5 MHz. Hence, we choose $L=5$. The vector t_l of the sampling instants inside a period T is then selected to be $t_l=[0\ \ 2d\ \ 3d\ \ 5d\ \ 6d]$. Step 4 of the proposed algorithm confirms the proper choice of T, d, and t_l. Note that $\mathbf{Z}_S$ is thus:

$$\mathbf{Z}_S=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\\1&1&1\end{bmatrix},$$

or in other words $\operatorname{rank}[\mathbf{Z}_S]=1$, which confirms that the achieved average sampling rate of $f_{S_{ave}}=L/T=5$ MHz is the optimal rate, a 29% lower average sampling rate than the Landau rate for signal reconstruction. The results presented below confirm the feasibility of the proposed approach. Figure 5 shows the power spectrum density of the input and output signals. Figure 6 shows the magnitude of the spectrum of filtering error when Filter A

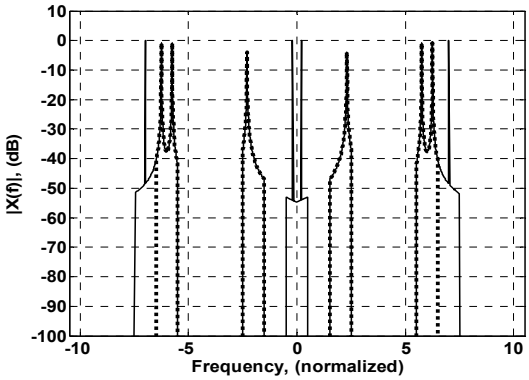

Figure 5. Power spectrum density of input and output signals: a) solid line – input signal; b) dotted line – output signal.

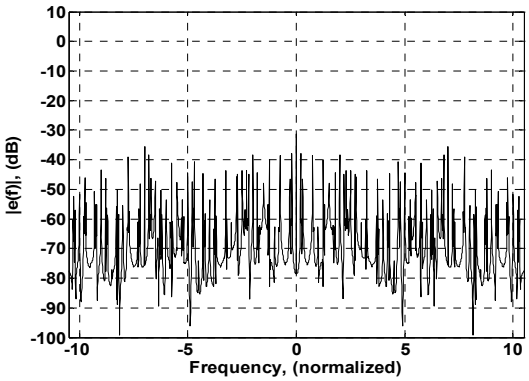

Figure 6. Magnitude of the spectrum of filtering error – Filter A scheme using lowpass FIR filter.

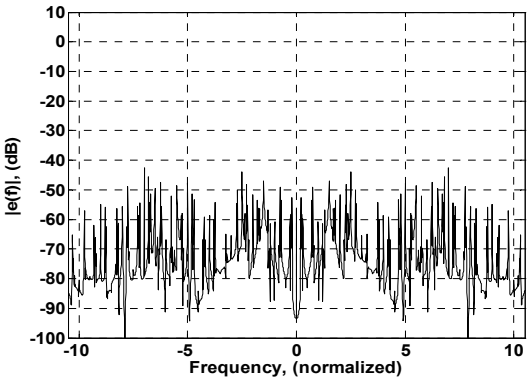

Figure 7. Magnitude of the spectrum of filtering error – Filter B scheme using multiband FIR filters.

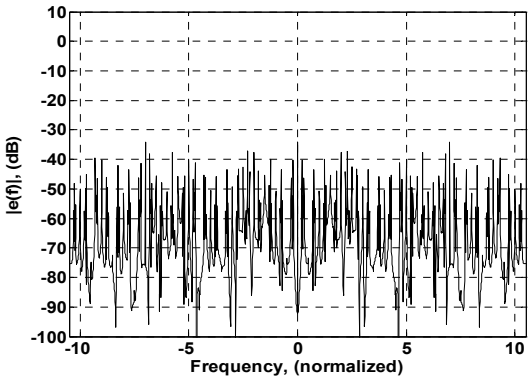

Figure 8. Magnitude of the spectrum of filtering error – Filter A scheme using lowpass IIR filter.

scheme has been constructed using a lowpass equiripple FIR filter [13] of order 221, which has a transition band of 0.25/*T*, 0.032 dB passband ripple and -48.75 dB stopband attenuation. Figure 7 shows the magnitude of the spectrum of filtering error when Filter B scheme has been realized, using multiband Chebyshev FIR filters [14] of order 221, with transition band of 0.25/*T*, 0.03 dB passband ripple and -57.8 dB stopband attenuation. Figure 8 shows the magnitude of the spectrum of the filtering error when Filter A scheme has been constructed using a lowpass IIR filter of order 10, acquired from the lowpass FIR filter used above through balanced model reduction [15]. The IIR filter has the same bandwidth and transition band as the lowpass FIR filter, and has 0.095 dB passband ripple and -47.9 dB stopband attenuation. All three realizations produced almost identical results. However, Filter A scheme has the advantage of using one lowpass filter, whereas Filter B scheme uses three different multiband filters (two pairs of the branches of Filter B structure happen to use the same filters).

Example 3 – Bandpass filtering of a multiband signal

Consider a random test signal which has the following spectral support frequency division:

$$SSF_{in} = (-23.5,-22.5)\cup(-16.5,-15.5)\cup(-9.5,-5.5)\cup\ldots$$
$$\ldots\cup(-2.5,-1.5)\cup(-0.5,0.5)\cup(1.5,2.5)\cup(5.5,9.5)\cup\ldots$$
$$\ldots\cup(15.5,16.5)\cup(22.5,23.5)\ \text{MHz}.$$

Let the filter passbands be :

$$SSF_{out} = (-6.5,-5.5)\cup(-2.5,-1.5)\cup(1.5,2.5)\cup(5.5,6.5)\ \text{MHz}.$$

The Nyquist uniform sampling rate for this signal is $f_{Nyquist} = 2f_{\max} = 2*23.5 = 47$ MHz. The Landau rate, or the minimum average sampling rate for sampling and reconstruction of the input signal, is $f_{Landau} = 15$ MHz. Initiating the algorithm described in section IV, we choose $T = 1$ μs. Hence, the input and output spectral supports can be represented as:

$$F_{in} = [-23\ -16\ -9\ -8\ -7\ -6\ -2\ 0\ 2\ 6\ 7\ 8\ 9\ 16\ 23]$$

and $F_{out} = [-6\ \ -2\ \ 2\ \ 6]$. We choose $d = 1/16$ μs, as uniform sampling of the input signal with rate $f_d = 16$ MHz does not induce aliasing in the passbands. Also, uniform sampling of the input signal at such rate shows that the total bandwidth of the spectral components in the interval $[-8,8)$ MHz is 8 MHz. Hence, we choose $L = 8$. The vector t_l of the sampling instants inside a period T is then selected to be $t_l = [0\ \ 2d\ \ 5d\ \ 6d\ \ 7d\ \ 10d\ \ 11d\ \ 15d]$. Step 4 of the proposed algorithm confirms the proper choice of T, d, and t_l. The achieved average sampling rate of $f_{S_{ave}} = L/T = 8$ MHz is the optimal rate, a 47% lower average sampling rate than the Landau rate for signal reconstruction. Figure 9 shows the power spectrum density of the input and output signals. Figure 10 shows the magnitude of the spectrum of filtering error when Filter A scheme has been constructed using a lowpass

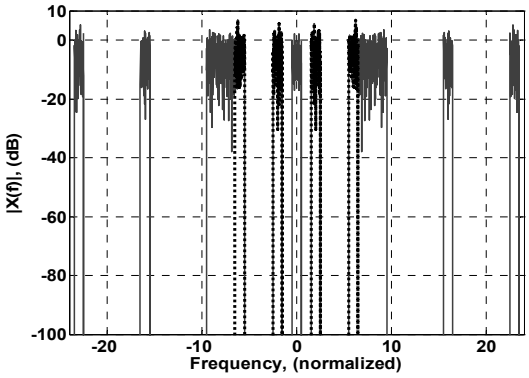

Figure 9. Power spectrum density of input and output signals: a) solid line – input signal; b) dotted line – output signal.

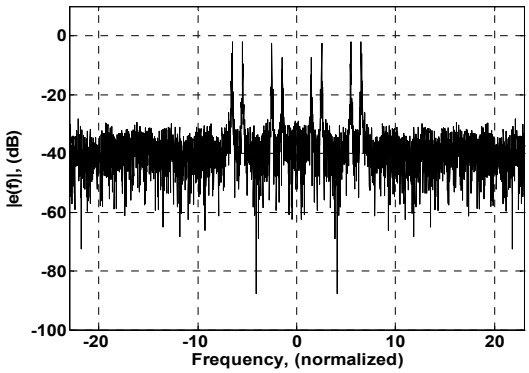

Figure 10. Magnitude of the spectrum of filtering error – Filter A scheme using lowpass FIR filter.

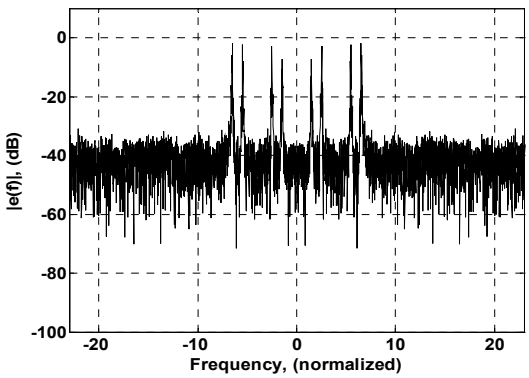

Figure 11. Magnitude of the spectrum of filtering error – Filter B scheme using multiband FIR filters.

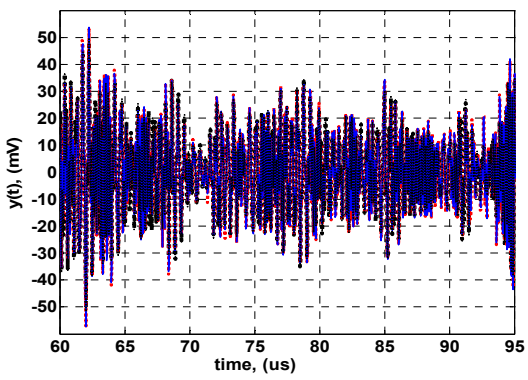

Figure 12. Output signals: a) dotted black line – ideal output; b) blue line – Filter A; c) dotted red line – Filter B.

equiripple FIR filter [13] of order 353, which has a transition band of 0.3/*T*, 0.05 dB passband ripple and -44.65 dB stopband attenuation. Figure 11 shows the magnitude of the spectrum of filtering error when Filter B scheme has been realized, using multiband Chebyshev FIR filters [14] of order 353, with transition band of 0.3/*T*, 0.052 dB passband ripple and -47.8 dB stopband attenuation. Both realizations produced almost identical outputs, as shown in Figure 12. (Note that the peak errors in the spectral plots are due to the wider transition bands of the filters.)

VII. Conclusions

The possibility of filtering of band-limited signals by means of linear filter with one or more stopbands using lower than Landau sampling rates was discussed. We have shown that in many cases signals can be filtered at such low rates. When lowpass filtering of baseband signal is needed, the examined approach does not offer an improvement over classical DSP for uniformly sampled signals. However, when signals with passband or multiband spectra are processed, filtering using PNS can be conducted using substantially lower sampling rates. The proposed method is thus applicable in high frequency applications, or where high-speed data processing is not cost-effective, such as radio communications, astrology, geology, etc.

Necessary and sufficient conditions for filtering have been derived, and a procedure for designing PNS schemes which facilitates the proposed approach has been proposed. Two different filter realizations have been developed and compared – the first, Filter A, uses one lowpass filter, whereas the second, Filter B, uses up to *L* different passband or multiband filters. The experimental results have confirmed the feasibility of the approach, and suggest that the quality of filtering highly depends on the quality of the real FIR or IIR subfilters that are used in practice due to the requirement of having sharp transition bands.

There are still a number of tasks that need to be tackled. The time-varying nature of such filtering system using PNS requires a more rigorous analysis and optimization to fully describe the filter's characteristics, such as error analysis, sensitivity analysis, optimal PNS scheme analysis, etc., which consequently could ease the requirements on the system and its subfilters. Another goal is to transfer the analysis to two-dimensional and three-dimensional signal processing, such as in electrocardiogramming (ECG), magnetic resonance imaging (MRI), etc, where sampling at lower rates will be highly cost-effective.

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