An Algebraic Method for Estimating the Fundamental Matrix with Rank Constraint

WANG Yuanbin
School of Information Science and Engineering, Northeastern University, China
Email: wangyuanbin@ise.neu.edu.cn

ZHANG Bin and YAO Tianshun
School of Information Science and Engineering, Northeastern University, China
Email: zhangbin@ise.neu.edu.cn

***Abstract*—The fundamental matrix captures the intrinsic geometric properties of two images of a same 3D scene. It should be of rank two for all the epipolar lines to intersect in a unique epipole. Traditional methods of enforcing the rank two property of the matrix are to parameterize the fundamental matrix during the estimation. This usually results in a system of nonlinear multivariable polynomial equations of higher degree. The solution of which is then hand over to some numerical techniques. Numerical precision analysis and convergence proof of these solutions are needed but neglected. This paper studies the structure of the typical nonlinear multivariable polynomial equations encountered in the fundamental matrix estimation with rank constraint. An algebraic method is presented to solve this type of equations. The method is based on the classical Lagrange multipliers method. After careful transformations of the problem, we reduce the problem to the solution of a single variable polynomial equation.**

***Index Terms*—computer vision, epipolar geometry, fundamental matrix, nonlinear multivariable equations**

I. Introduction

The study of the geometric and algebraic constraints that hold among multiple views of the same 3D scene is called the geometry of multiple views [7]. Specifically, the epipolar geometry is the geometry relating two images of a 3D scene captured from two pinhole cameras from distinct positions. In epipolar geometry, the observed 3D point P and the baseline joining the optical centers O and O' of the two cameras define the epipolar plane. The projections of point P through projection centers O and O' onto the two image planes of the cameras are p and p' respectively. The two points e and e' at which the baseline intersects the image planes are called the epipoles of the two cameras. Let the homogeneous coordinates of p and p' be $(x, y, 1)$ and $(x', y', 1)$ respectively, it is well known that there is a 3×3 matrix F such that

$$\begin{pmatrix} x & y & 1 \end{pmatrix} F \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = 0. \qquad (1)$$

This matrix is called the fundamental matrix and it represents the epipolar constraint that holds between two views of the same 3D scene. It is defined only up to a scale and has rank 2. So the fundamental matrix has only seven independent parameters. The epipolar geometry is useful in 3D scene reconstruction. It can also be used to limit the search space for point correspondences in stereo vision and to constrain the set of possible motions in camera motion analysis.

The computation of the fundamental matrix given a set of point correspondences between two images has been studied for decades. The fundamental matrix should be of rank two for all the epipolar lines to intersect in a unique epipole. Most of the traditional methods for estimating the parameters of the fundamental matrix usually ignore the rank two property of the matrix at the first step of derivation and then replace the computed matrix with a new matrix to enforce this constraint. So the resulting matrix is only an approximation to an ideal fundamental matrix that captures the accurate geometric relationship of the two views. A few methods try to estimate the fundamental matrix with rank-2 constraint through nonlinear minimization in parameter space [1, 2, 3, 7, 10, 14, 17]. The resulting objective functions to minimize are nonlinear. The minimization technique is usually based on Newton-Raphson method which is time consuming and may not always converge.

This paper presents an effective method to solve the typical constrained minimization problem encountered in the fundamental matrix estimation with rank-2 constraint. In section 2 we present a brief overview of the previous methods for the fundamental matrix estimation and introduce the constrained minimization problem. In section 3 we present the proposed effective method to solve the typical minimization problem. The method is based on the classical method of Lagrange multipliers for solving equality constraint problems. Section 4 is the conclusion.

Manuscript received August 1, 2009; revised September 1, 2009; accepted October 1, 2009.

WANG YuanBin is with the Institute of Computer Application Technology, Northeastern University, Shenyang, 110004, PRC. E-mail: wangyuanbin@ise.neu.edu.cn.

doi:10.4304/jcp.5.7.1027-1037

II. STATEMENT OF THE PROBLEM

Longuet-Higgins first proposed the eight-point algorithm to compute the essential matrix for calibrated cameras in 1981 [11]. Faugeras and Hartley applied the eight-point algorithm to derive the fundamental matrix in [4, 5]. Since that time, a large number of methods have been developed to estimate the fundamental matrix. Those methods can be classified into three categories: linear method, iterative method and robust method [1]. The input to those methods is a set of point correspondences between the two images.

Linear methods are simple, fast and easy to implement. They provide good estimate when noise level is low and there is no outliers of point correspondences. They are often utilized in more complicated estimation schemas. Let the fundamental matrix F be

$$F = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} \qquad (2)$$

and the set of point correspondences between the two images be

$$p_i = \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix} \leftrightarrow p_i' = \begin{pmatrix} x_i' \\ y_i' \\ 1 \end{pmatrix}, i = 1, 2, 3, \cdots. \qquad (3)$$

From the epipolar constraint (1), a linear equation in the nine parameters of the fundamental matrix is obtained for each point match

$$t_i f = 0, \qquad (4)$$

where

$$t_i = \left(x_i x_i', x_i y_i', x_i, y_i x_i', y_i y_i', y_i, x_i', y_i', 1 \right) \qquad (5)$$

and

$$f = \left(f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33} \right)^T. \qquad (6)$$

The seven-point method has the advantage of computing the fundamental matrix using only seven point correspondences [7, 15, 16]. In this case, the solution to the system of linear equations

$$\begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_7 \end{pmatrix} f = 0 \qquad (7)$$

is a 2-dimensional space of the form

$$F = \alpha F_1 + (1-\alpha) F_2, \qquad (8)$$

where α is a scalar variable. Since the fundamental matrix is rank deficient, setting

$$\det(\alpha F_1 + (1-\alpha) F_2) = 0 \qquad (9)$$

will result in a cubic polynomial equation in α. Substituting back the real solutions (discarding the complex solutions) of (9) into (8) gives one or three possible solutions of the fundamental matrix.

The most notable and the simplest linear method of computing the fundamental matrix is the eight-pint algorithm. Suppose that eight point correspondences are given. We can set up a system of homogeneous linear equations of the form

$$\begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_8 \end{pmatrix} f = 0. \qquad (10)$$

By setting one of the parameters of F to 1, (10) becomes a system of eight non-homogeneous linear equations with eight unknowns. The solution of the resulting equations is sufficient for estimating the fundamental matrix [4, 5, 6, 7]. When more than eight points are available, F can be estimated using linear least squares by minimizing the residual of (1) for each point correspondences

$$\underset{F}{\text{minimize}} \sum_{i=1}^{n} \left(t_i f \right)^2. \qquad (11)$$

The iterative methods start from an initial estimate of the fundamental matrix then iterate to find the final F that minimizes algebraic error. Luong and Faugeras proposed an iterative method that minimize the following equation [12]

$$\underset{F}{\text{minimize}} \sum_{i} w_i (t_i f)^2, \qquad (12)$$

where

$$w_i = \left(\frac{1}{l_1^2 + l_2^2} + \frac{1}{l_1'^2 + l_2'^2} \right), \qquad (13)$$

$$F p_i' = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}, F^T p_i = \begin{pmatrix} l_1' \\ l_2' \\ l_3' \end{pmatrix}. \qquad (14)$$

Robust methods of fundamental matrix estimation are needed when noise level in image point measurements is high and incorrect point correspondences are present. M-Estimators try to reduce the effect of point noise and outliers by applying weight functions [15, 16]. They are based on solving a following weighted least squares problem

$$\underset{F}{\text{minimize}} \sum_i w_i (t_i f)^2 , \tag{15}$$

where w_i is a weight function. There have been many weight functions proposed for the M-Estimator method. A common weight function is proposed by Huber as [8]

$$w_i = \begin{cases} 1 & |r_i| \le \sigma \\ \dfrac{\sigma}{|r_i|} & \sigma < |r_i| \le 3\sigma \\ 0 & 3\sigma < |r_i| \end{cases} , \tag{16}$$

where r_i is the residual $t_i f$ of each point.

Since there is no efficient method to solve nonlinear multivariable polynomial equations of higher degree, the minimization of (11), (12), and (15) usually ignores the rank-2 constraint of the fundamental matrix during the estimation. The obtained matrix is transformed into a new rank-2 matrix through singular value decomposition if the rank-2 constraint is needed.

To enforce the rank-2 property of the matrix during the estimation, we need to solve a constrained minimization problem of the form

$$\begin{aligned} &\underset{F}{\text{minimize}} \sum_i w_i (t_i f)^2 \\ &\text{subject to} \quad \det(F) = 0. \end{aligned} \tag{17}$$

Traditional methods of solving (17) are to parameterize the matrix F and then convert the constrained minimization problem to an unconstrained minimization problem. One method of parameterization is to represent one column of the matrix F as a linear combination of the other two columns, such as

$$F = \begin{pmatrix} a & b & \alpha a + \beta b \\ c & d & \alpha c + \beta d \\ e & f & \alpha e + \beta f \end{pmatrix}. \tag{18}$$

The main disadvantage is that it does not work when the first two columns are linearly dependent. Such singularities can be solved by switching to one of the alternative parameterizations during the minimization. Another method of parameterization of F uses both of the two epipoles $(\alpha, \beta, -1)$ and $(\alpha^{'}, \beta^{'}, -1)$ as parameters

$$F = \begin{pmatrix} a & b & \alpha a + \beta b \\ c & d & \alpha c + \beta d \\ \alpha^{'} a + \beta^{'} c & \alpha^{'} b + \beta^{'} d & f_{33} \end{pmatrix} \tag{19}$$

where $f_{33} = \alpha^{'} (\alpha a + \beta b) + \beta^{'} (\alpha c + \beta d)$. We can choose different two rows and two columns to use as the basis to avoid singularities.

After parameterizing the fundamental matrix and then eliminating dependent variables, the constrained optimization problem (17) is reduced to an unconstrained optimization problem. Traditional method to solve this problem is based on numerical Newton-Raphson iteration technique. The solution is time consuming and is not accurate enough sometimes. There is also no guaranty of convergence for this method. We will propose an algebraic method to solve this kind of constraint optimization problem in the next section.

III. THE PROPOSED METHOD

In this section, we propose an effective and algebraic method to solve the typical constraint minimization problem of the type (17) encountered in the fundamental matrix estimation with rank-2 constraint.

Classically, there are two methods to reduce the equality constraint optimization problem to an unconstrained optimization problem. One method which is widely adopted in the fundamental matrix estimation algorithms is to eliminate some dependent variables by substituting them into the objective function. This will first result in a system of polynomial equations of third degree. According to Bezout's theorem, for 8 polynomial equations of the third degree, there are in general at most 3^8=6561 common solutions if the 8 polynomial equations do not contain common factors. So the structure of the solutions is rather complicated.

The other method of transforming the equality constraint optimization problem into unconstraint optimization problem is by means of the Lagrange multipliers. Although the method of Lagrange multipliers is not very practical for solving general equality constraint minimization problems, we show that this method can be applied to solve the problem (17) effectively.

A. The basic polynomial equations

The condition that $\det(F) = 0$ in (17) means that one column of F is a linear combination of the other two columns. Without loss of generality, we suppose that

$$\begin{aligned} uf_{11} + vf_{12} + f_{13} = 0, \\ uf_{21} + vf_{22} + f_{23} = 0, \\ uf_{31} + vf_{32} + f_{33} = 0, \end{aligned} \tag{20}$$

where u and v are scalars that are not all zero.

Let us define G_1, G_2, and G_3 as

$$\begin{aligned} G_1 = uf_{11} + vf_{12} + f_{13}, \\ G_2 = uf_{21} + vf_{22} + f_{23}, \\ G_3 = uf_{31} + vf_{32} + f_{33}. \end{aligned} \tag{21}$$

Now we construct the Lagrangian function

$$H = \sum_{i=1}^{n} w_i (t_i f)^2 - \alpha G_1 - \beta G_2 - \lambda G_3 . \tag{22}$$

It is well known from the theory of optimization that the solution of this unconstrained problem (22) contains the solution of the original constrained problem (17). Applying the Lagrange condition yields

$$\begin{aligned}\frac{\partial H}{\partial f_{11}} &= \left(\sum_{i=1}^{n} 2w_i x_i x_i^{'} t_i\right) f - \alpha u = 0,\\ \frac{\partial H}{\partial f_{12}} &= \left(\sum_{i=1}^{n} 2w_i x_i y_i^{'} t_i\right) f - \alpha v = 0,\\ \frac{\partial H}{\partial f_{13}} &= \left(\sum_{i=1}^{n} 2w_i x_i t_i\right) f - \alpha = 0,\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{\partial H}{\partial f_{21}} &= \left(\sum_{i=1}^{n} 2w_i y_i x_i^{'} t_i\right) f - \beta u = 0,\\ \frac{\partial H}{\partial f_{22}} &= \left(\sum_{i=1}^{n} 2w_i y_i y_i^{'} t_i\right) f - \beta v = 0,\\ \frac{\partial H}{\partial f_{23}} &= \left(\sum_{i=1}^{n} 2w_i y_i t_i\right) f - \beta = 0,\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{\partial H}{\partial f_{31}} &= \left(\sum_{i=1}^{n} 2w_i x_i^{'} t_i\right) f - \lambda u = 0,\\ \frac{\partial H}{\partial f_{32}} &= \left(\sum_{i=1}^{n} 2w_i y_i^{'} t_i\right) f - \lambda v = 0,\\ \frac{\partial H}{\partial f_{33}} &= \left(\sum_{i=1}^{n} 2w_i t_i\right) f - \lambda = 0.\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{\partial H}{\partial \alpha} &= uf_{11} + vf_{12} + f_{13} = 0,\\ \frac{\partial H}{\partial \beta} &= uf_{21} + vf_{22} + f_{23} = 0,\\ \frac{\partial H}{\partial \lambda} &= uf_{31} + vf_{32} + f_{33} = 0.\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{\partial H}{\partial u} &= \alpha f_{11} + \beta f_{21} + \lambda f_{31} = 0,\\ \frac{\partial H}{\partial v} &= \alpha f_{12} + \beta f_{22} + \lambda f_{32} = 0.\end{aligned} \tag{27}$$

Equation (23), (24), and (25) can be organized as a system of linear equations of the following form

$$\begin{pmatrix} \sum_{i=1}^{n} 2w_i x_i x_i^{'} t_i \\ \sum_{i=1}^{n} 2w_i x_i y_i^{'} t_i \\ \sum_{i=1}^{n} 2w_i x_i t_i \\ \sum_{i=1}^{n} 2w_i y_i x_i^{'} t_i \\ \sum_{i=1}^{n} 2w_i y_i y_i^{'} t_i \\ \sum_{i=1}^{n} 2w_i y_i t_i \\ \sum_{i=1}^{n} 2w_i x_i^{'} t_i \\ \sum_{i=1}^{n} 2w_i y_i^{'} t_i \\ \sum_{i=1}^{n} 2w_i t_i \end{pmatrix} \begin{pmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{pmatrix} = \begin{pmatrix} \alpha u \\ \alpha v \\ \alpha \\ \beta u \\ \beta v \\ \beta \\ \lambda u \\ \lambda v \\ \lambda \end{pmatrix}. \tag{28}$$

Let us denote the coefficient matrix in (28) as **Φ**. Let D be the determinant of the coefficient matrix **Φ**. Let D_{ij} be cofactors corresponding to row i and column j of the determinant of the coefficient matrix **Φ**. Then the solution of (28) is

$$\begin{aligned}f_{11} = (&D_{11}\alpha u + D_{12}\alpha v + D_{13}\alpha + D_{14}\beta u + D_{15}\beta v\\ &+ D_{16}\beta + D_{17}\lambda u + D_{18}\lambda v + D_{19}\lambda) / D,\end{aligned} \tag{29}$$

$$\begin{aligned}f_{12} = (&D_{21}\alpha u + D_{22}\alpha v + D_{23}\alpha + D_{24}\beta u + D_{25}\beta v\\ &+ D_{26}\beta + D_{27}\lambda u + D_{28}\lambda v + D_{29}\lambda) / D,\end{aligned} \tag{30}$$

$$\begin{aligned}f_{13} = (&D_{31}\alpha u + D_{32}\alpha v + D_{33}\alpha + D_{34}\beta u + D_{35}\beta v\\ &+ D_{36}\beta + D_{37}\lambda u + D_{38}\lambda v + D_{39}\lambda) / D,\end{aligned} \tag{31}$$

$$\begin{aligned}f_{21} = (&D_{41}\alpha u + D_{42}\alpha v + D_{43}\alpha + D_{44}\beta u + D_{45}\beta v\\ &+ D_{46}\beta + D_{47}\lambda u + D_{48}\lambda v + D_{49}\lambda) / D,\end{aligned} \tag{32}$$

$$\begin{aligned}f_{22} = (&D_{51}\alpha u + D_{52}\alpha v + D_{53}\alpha + D_{54}\beta u + D_{55}\beta v\\ &+ D_{56}\beta + D_{57}\lambda u + D_{58}\lambda v + D_{59}\lambda) / D,\end{aligned} \tag{33}$$

$$\begin{aligned}f_{23} = (&D_{61}\alpha u + D_{62}\alpha v + D_{63}\alpha + D_{64}\beta u + D_{65}\beta v\\ &+ D_{66}\beta + D_{67}\lambda u + D_{68}\lambda v + D_{69}\lambda) / D,\end{aligned} \tag{34}$$

$$f_{31} = (D_{71}\alpha u + D_{72}\alpha v + D_{73}\alpha + D_{74}\beta u + D_{75}\beta v + D_{76}\beta + D_{77}\lambda u + D_{78}\lambda v + D_{79}\lambda) / D, \quad (35)$$

$$f_{32} = (D_{81}\alpha u + D_{82}\alpha v + D_{83}\alpha + D_{84}\beta u + D_{85}\beta v + D_{86}\beta + D_{87}\lambda u + D_{88}\lambda v + D_{89}\lambda) / D, \quad (36)$$

$$f_{33} = (D_{91}\alpha u + D_{92}\alpha v + D_{93}\alpha + D_{94}\beta u + D_{95}\beta v + D_{96}\beta + D_{97}\lambda u - D_{98}\lambda v + D_{99}\lambda) / D. \quad (37)$$

Substituting the solution of f into equations (26) and (27), we obtain 5 polynomial equations of degree three

$$\begin{aligned}\Gamma_1 &= D_{11}\alpha u^2 + ((D_{21}+D_{12})v + (D_{13}+D_{31}))\alpha u \\ &+ (D_{22}v^2 + (D_{32}+D_{23})v + D_{33})\alpha + \\ &D_{41}\beta u^2 + ((D_{51}+D_{42})v + (D_{43}+D_{61}))\beta u + \\ &(D_{52}v^2 + (D_{62}+D_{53})v + D_{63})\beta + \\ &D_{71}\lambda u^2 + ((D_{81}+D_{72})v + (D_{73}+D_{91}))\lambda u + \\ &(D_{82}v^2 + (D_{92}+D_{83})v + D_{93})\lambda = 0,\end{aligned} \quad (38)$$

$$\begin{aligned}\Gamma_2 &= D_{14}\alpha u^2 + ((D_{24}+D_{15})v + (D_{34}+D_{16}))\alpha u \\ &+ (D_{25}v^2 + (D_{35}+D_{26})v + D_{36})\alpha + \\ &D_{44}\beta u^2 + ((D_{54}+D_{45})v + (D_{64}+D_{46}))\beta u + \\ &(D_{55}v^2 + (D_{65}+D_{56})v + D_{66})\beta + \\ &D_{74}\lambda u^2 + ((D_{84}+D_{75})v + (D_{94}+D_{76}))\lambda u + \\ &(D_{85}v^2 + (D_{95}+D_{86})v + D_{96})\lambda = 0,\end{aligned} \quad (39)$$

$$\begin{aligned}\Gamma_3 &= D_{17}\alpha u^2 + ((D_{27}+D_{18})v + (D_{37}+D_{19}))\alpha u \\ &+ (D_{28}v^2 + (D_{38}+D_{29})v + D_{39})\alpha + \\ &D_{47}\beta u^2 + ((D_{57}+D_{48})v + (D_{67}+D_{49}))\beta u + \\ &(D_{58}v^2 + (D_{68}+D_{59})v + D_{69})\beta + \\ &D_{77}\lambda u^2 + ((D_{87}+D_{78})v + (D_{97}+D_{79}))\lambda u + \\ &(D_{88}v^2 + (D_{98}+D_{89})v + D_{99})\lambda = 0,\end{aligned} \quad (40)$$

$$\begin{aligned}\Gamma_4 &= D_{11}\alpha^2 u + (D_{21}v + D_{31})\alpha^2 + \\ &(D_{41}+D_{14})\alpha\beta u + \\ &((D_{51}+D_{24})v + D_{61} + D_{34})\alpha\beta + (D_{71}+D_{17}) \\ &\alpha\lambda u + ((D_{81}+D_{27})v + D_{91} + D_{37})\alpha\lambda + \\ &D_{44}\beta^2 u + (D_{54}v + D_{64})\beta^2 + (D_{74}+D_{47})\beta\lambda u \\ &+ ((D_{84}+D_{57})v + D_{94} + D_{67})\beta\lambda + \\ &D_{77}\lambda^2 u + (D_{87}v + D_{97})\lambda^2 = 0,\end{aligned} \quad (41)$$

$$\begin{aligned}\Gamma_5 &= D_{12}\alpha^2 u + (D_{22}v + D_{32})\alpha^2 + (D_{42}+D_{15}) \\ &\alpha\beta u + ((D_{52}+D_{25})v + D_{62} + D_{35})\alpha\beta + \\ &(D_{72}+D_{18})\alpha\lambda u + ((D_{82}+D_{28})v + D_{92} + D_{38})\alpha\lambda \\ &+ D_{45}\beta^2 u + (D_{55}v + D_{65})\beta^2 + (D_{75}+D_{48})\beta\lambda u \\ &+ ((D_{85}+D_{58})v + D_{95} + D_{68})\beta\lambda + \\ &D_{78}\lambda^2 u + (D_{88}v + D_{98})\lambda^2 = 0.\end{aligned} \quad (42)$$

Equations (38) to (42) are of the third degree. Classical algebraic methods to solve polynomial equations (38) to (42) include elimination of the unknowns through resultants or transforming the set of polynomial equations into Gröbner basis. According to Bezout's theorem, for 5 polynomial equations of third degree, there are in general 3^5=243 common solutions. The computational complexity is also quite high. We now propose a simple and clear method to solve this type of equations.

B. *The equations with two variables*

We first transform the set of polynomial equations in 5 variables derived previously into an equation in two variables in this subsection. For presentation simplicity, we rewrite (38), (39), and (40) as

$$\begin{aligned} b_1\alpha + b_2\beta + b_3\lambda = 0 \\ c_1\alpha + c_2\beta + c_3\lambda = 0 \\ d_1\alpha + d_2\beta + d_3\lambda = 0 \end{aligned} \; . \quad (43)$$

From (43), we have

$$\begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = 0 \, . \quad (44)$$

Since α, β, and λ are not all zero, the determinant of the coefficient matrix in (44) must be zero. That is

$$\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0 \, . \quad (45)$$

It is easy to see that (45) is a polynomial equation in two variables u and v. Equation (45) is a polynomial of the sixth degree.

C. *The equation with a single variable*

We transform the set of polynomial equations in 5 variables derived previously into one polynomial equation in a single variable in this subsection. For presentation simplicity, we denote (38), (39), (40), (41), and (42) as

$$\begin{aligned}\Gamma_1 &= A_1\alpha u^2 + A_2\alpha u + A_3\alpha + A_4\beta u^2 + \\ &A_5\beta u + A_6\beta + A_7\lambda u^2 + A_8\lambda u + A_9\lambda = 0\end{aligned}, \quad (46)$$

$$\Gamma_2 = B_1\alpha u^2 + B_2\alpha u + B_3\alpha + B_4\beta u^2 + B_5\beta u + B_6\beta + B_7\lambda u^2 + B_8\lambda u + B_9\lambda = 0, \quad (47)$$

$$\Gamma_3 = C_1\alpha u^2 + C_2\alpha u + C_3\alpha + C_4\beta u^2 + C_5\beta u + C_6\beta + C_7\lambda u^2 + C_8\lambda u + C_9\lambda = 0, \quad (48)$$

$$\Gamma_4 = S_1\alpha^2 u + S_2\alpha^2 + S_3\alpha\beta u + S_4\alpha\beta + S_5\alpha\lambda u + S_6\alpha\lambda + S_7\beta^2 u + S_8\beta^2 + S_9\beta\lambda u + S_{10}\beta\lambda + S_{11}\lambda^2 u + S_{12}\lambda^2 = 0, \quad (49)$$

$$\Gamma_5 = T_1\alpha^2 u + T_2\alpha^2 + T_3\alpha\beta u + T_4\alpha\beta + T_5\alpha\lambda u + T_6\alpha\lambda + T_7\beta^2 u + T_8\beta^2 + T_9\beta\lambda u + T_{10}\beta\lambda + T_{11}\lambda^2 u + T_{12}\lambda^2 = 0. \quad (50)$$

From (46), we can construct 6 polynomial equations of the following form

$$\alpha^2\Gamma_1 = A_1\alpha^3 u^2 + A_2\alpha^3 u + A_3\alpha^3 + A_4\alpha^2\beta u^2 + A_5\alpha^2\beta u + A_6\alpha^2\beta + A_7\alpha^2\lambda u^2 + A_8\alpha^2\lambda u + A_9\alpha^2\lambda = 0, \quad (51)$$

$$\alpha\beta\Gamma_1 = A_1\alpha^2\beta u^2 + A_2\alpha^2\beta u + A_3\alpha^2\beta + A_4\alpha\beta^2 u^2 + A_5\alpha\beta^2 u + A_6\alpha\beta^2 + A_7\alpha\beta\lambda u^2 + A_8\alpha\beta\lambda u + A_9\alpha\beta\lambda = 0, \quad (52)$$

$$\alpha\lambda\Gamma_1 = A_1\alpha^2\lambda u^2 + A_2\alpha^2\lambda u + A_3\alpha^2\lambda + A_4\alpha\beta\lambda u^2 + A_5\alpha\beta\lambda u + A_6\alpha\beta\lambda + A_7\alpha\lambda^2 u^2 + A_8\alpha\lambda^2 u + A_9\alpha\lambda^2 = 0, \quad (53)$$

$$\beta^2\Gamma_1 = A_1\alpha\beta^2 u^2 + A_2\alpha\beta^2 u + A_3\alpha\beta^2 + A_4\beta^3 u^2 + A_5\beta^3 u + A_6\beta^3 + A_7\beta^2\lambda u^2 + A_8\beta^2\lambda u + A_9\beta^2\lambda = 0, \quad (54)$$

$$\beta\lambda\Gamma_1 = A_1\alpha\beta\lambda u^2 + A_2\alpha\beta\lambda u + A_3\alpha\beta\lambda + A_4\beta^2\lambda u^2 + A_5\beta^2\lambda u + A_6\beta^2\lambda + A_7\beta\lambda^2 u^2 + A_8\beta\lambda^2 u + A_9\beta\lambda^2 = 0, \quad (55)$$

$$\lambda^2\Gamma_1 = A_1\alpha\lambda^2 u^2 + A_2\alpha\lambda^2 u + A_3\alpha\lambda^2 + A_4\beta\lambda^2 u^2 + A_5\beta\lambda^2 u + A_6\beta\lambda^2 + A_7\lambda^3 u^2 + A_8\lambda^3 u + A_9\lambda^3 = 0. \quad (56)$$

From (49), we can construct 6 polynomial equations of the following form

$$\alpha\Gamma_4 = S_1\alpha^3 u + S_2\alpha^3 + S_3\alpha^2\beta u + S_4\alpha^2\beta + S_5\alpha^2\lambda u + S_6\alpha^2\lambda + S_7\alpha\beta^2 u + S_8\alpha\beta^2 + S_9\alpha\beta\lambda u + S_{10}\alpha\beta\lambda + S_{11}\alpha\lambda^2 u + S_{12}\alpha\lambda^2 = 0, \quad (57)$$

$$\beta\Gamma_4 = S_1\alpha^2\beta u + S_2\alpha^2\beta + S_3\alpha\beta^2 u + S_4\alpha\beta^2 + S_5\alpha\beta\lambda u + S_6\alpha\beta\lambda + S_7\beta^3 u + S_8\beta^3 + S_9\beta^2\lambda u + S_{10}\beta^2\lambda + S_{11}\beta\lambda^2 u + S_{12}\beta\lambda^2 = 0, \quad (58)$$

$$\lambda\Gamma_4 = S_1\alpha^2\lambda u + S_2\alpha^2\lambda + S_3\alpha\beta\lambda u + S_4\alpha\beta\lambda + S_5\alpha\lambda^2 u + S_6\alpha\lambda^2 + S_7\beta^2\lambda u + S_8\beta^2\lambda + S_9\beta\lambda^2 u + S_{10}\beta\lambda^2 + S_{11}\lambda^3 u + S_{12}\lambda^3 = 0, \quad (59)$$

$$\alpha u\Gamma_4 = S_1\alpha^3 u^2 + S_2\alpha^3 u + S_3\alpha^2\beta u^2 + S_4\alpha^2\beta u + S_5\alpha^2\lambda u^2 + S_6\alpha^2\lambda u + S_7\alpha\beta^2 u^2 + S_8\alpha\beta^2 u + S_9\alpha\beta\lambda u^2 + S_{10}\alpha\beta\lambda u + S_{11}\alpha\lambda^2 u^2 + S_{12}\alpha\lambda^2 u = 0, \quad (60)$$

$$\beta u\Gamma_4 = 0 = S_1\alpha^2\beta u^2 + S_2\alpha^2\beta u + S_3\alpha\beta^2 u^2 + S_4\alpha\beta^2 u + S_5\alpha\beta\lambda u^2 + S_6\alpha\beta\lambda u + S_7\beta^3 u^2 + S_8\beta^3 u + S_9\beta^2\lambda u^2 + S_{10}\beta^2\lambda u + S_{11}\beta\lambda^2 u^2 + S_{12}\beta\lambda^2 u, \quad (61)$$

$$\lambda u\Gamma_4 = 0 = S_1\alpha^2\lambda u^2 + S_2\alpha^2\lambda u + S_3\alpha\beta\lambda u^2 + S_4\alpha\beta\lambda u + S_5\alpha\lambda^2 u^2 + S_6\alpha\lambda^2 u + S_7\beta^2\lambda u^2 + S_8\beta^2\lambda u + S_9\beta\lambda^2 u^2 + S_{10}\beta\lambda^2 u + S_{11}\lambda^3 u^2 + S_{12}\lambda^3 u. \quad (62)$$

Similarly, we can construct another 18 polynomial equations from (47), (48), and (50). In this way, we obtain 30 polynomial equations. The 30 polynomial equations can be thought of as a system of homogeneous linear equations in 30 variables α^3u^2, α^3u, α^3, $\alpha^2\beta u^2$, $\alpha^2\beta u$, $\alpha^2\beta$, $\alpha\beta^2u^2$, $\alpha\beta^2u$, $\alpha\beta^2$, β^3u^2, β^3u, β^3, $\alpha^2\lambda u^2$, $\alpha^2\lambda u$, $\alpha^2\lambda$, $\alpha\beta\lambda u^2$, $\alpha\beta\lambda u$, $\alpha\beta\lambda$, $\beta^2\lambda u^2$, $\beta^2\lambda u$, $\beta^2\lambda$, $\alpha\lambda^2u^2$, $\alpha\lambda^2u$, $\alpha\lambda^2$, $\beta\lambda^2u^2$, $\beta\lambda^2u$, $\beta\lambda^2$, λ^3u^2, λ^3u, and λ^3. For this system of homogeneous linear equations to have nontrivial solutions, the determinant of the coefficient matrix **Ψ** has to be zero. This will result in a polynomial equation in single variable *v*. The explicit forms of the 30 row vectors of the coefficient matrix **Ψ** are

$$\begin{aligned}\Psi_1 = (&D_{11}, D_{21}v + D_{31}, 0, D_{41} + D_{14},\\ &(D_{51} + D_{24})v + D_{61} + D_{34}, 0, D_{71} + D_{17},\\ &(D_{81} + D_{27})v + D_{91} + D_{37}, 0, 0, 0, 0, D_{44},\\ &D_{54}v + D_{64}, 0, D_{74} + D_{47},\\ &(D_{84} + D_{57})v + D_{94} + D_{67},\\ &0, 0, 0, 0, D_{77}, D_{87}v + D_{97}, 0, 0, 0, 0, 0, 0, 0),\end{aligned} \tag{63}$$

$$\begin{aligned}\Psi_2 = (&D_{12}, D_{22}v + D_{32}, 0, D_{42} + D_{15},\\ &(D_{52} + D_{25})v + D_{62} + D_{35}, 0, D_{72} + D_{18},\\ &(D_{82} + D_{28})v + D_{92} + D_{38}, 0, 0, 0, 0, D_{45},\\ &D_{55}v + D_{65}, 0, D_{75} + D_{48},\\ &(D_{85} + D_{58})v + D_{95} + D_{68}, 0, 0, 0, 0,\\ &D_{78}, D_{88}v + D_{98}, 0, 0, 0, 0, 0, 0, 0),\end{aligned} \tag{64}$$

$$\begin{aligned}\Psi_3 = (&D_{11}, (D_{21} + D_{12})v + D_{13} + D_{31},\\ &D_{22}v^2 + (D_{32} + D_{23})v + D_{33}, D_{41},\\ &(D_{51} + D_{42})v + D_{43} + D_{61},\\ &D_{52}v^2 + (D_{62} + D_{53})v + D_{63}, 0, 0, 0, 0, 0, 0, 0, 0,\\ &0, 0, 0, 0, 0, 0, 0, D_{71}, (D_{81} + D_{72})v + D_{73} + D_{91},\\ &D_{82}v^2 + (D_{92} + D_{83})v + D_{93}, 0, 0, 0, 0, 0, 0),\end{aligned} \tag{65}$$

$$\begin{aligned}\Psi_4 = (&D_{14}, (D_{24} + D_{15})v + D_{16} + D_{34},\\ &D_{25}v^2 + (D_{35} + D_{26})v + D_{36}, D_{44},\\ &(D_{54} + D_{45})v + D_{46} + D_{64},\\ &D_{55}v^2 + (D_{65} + D_{56})v + D_{66}, 0, 0, 0, 0, 0, 0, 0, 0,\\ &0, 0, 0, 0, 0, 0, 0, D_{74}, (D_{84} + D_{75})v + D_{76} + D_{94},\\ &D_{85}v^2 + (D_{95} + D_{86})v + D_{96}, 0, 0, 0, 0, 0, 0),\end{aligned} \tag{66}$$

$$\begin{aligned}\Psi_5 = ((&D_{17}, (D_{27} + D_{18})v + D_{19} + D_{37},\\ &D_{28}v^2 + (D_{38} + D_{29})v + D_{39}, D_{47},\\ &(D_{57} + D_{48})v + D_{49} + D_{67},\\ &D_{58}v^2 + (D_{68} + D_{59})v + D_{69}, 0, 0, 0, 0, 0, 0, 0, 0,\\ &0, 0, 0, 0, 0, 0, 0, D_{77}, (D_{87} + D_{78})v + D_{79} + D_{97},\\ &D_{88}v^2 + (D_{98} + D_{89})v + D99, 0, 0, 0, 0, 0, 0),\end{aligned} \tag{67}$$

$$\begin{aligned}\Psi_6 = (&0, D_{11}, D_{21}v + D_{31}, 0, D_{41} + D_{14},\\ &(D_{51} + D_{24})v + D_{61} + D_{34}, 0, D_{71} + D_{17},\\ &(D_{81} + D_{27})v + D_{91} + D_{37}, 0, 0, 0, 0, D_{44},\\ &D_{54}v + D_{64}, 0, D_{74} + D_{47},\\ &(D_{84} + D_{57})v + D_{94} + D_{67}, 0, 0, 0, 0,\\ &D_{77}, D_{87}v + D_{97}, 0, 0, 0, 0, 0, 0),\end{aligned} \tag{68}$$

$$\begin{aligned}\Psi_7 = (&0, D_{12}, D_{22}v + D_{32}, 0, D_{42} + D_{15},\\ &(D_{52} + D_{25})v + D_{62} + D_{35}, 0, D_{72} + D_{18},\\ &(D_{82} + D_{28})v + D_{92} + D_{38}, 0, 0, 0, 0, D_{45},\\ &D_{55}v + D_{65}, 0, D_{75} + D_{48},\\ &(D_{85} + D_{58})v + D_{95} + D_{68}, 0, 0, 0, 0, D_{78},\\ &D_{88}v + D_{98}, 0, 0, 0, 0, 0, 0),\end{aligned} \tag{69}$$

$$\begin{aligned}\Psi_8 = (&0, 0, 0, D_{11}, D_{21}v + D_{31}, 0, D_{41} + D_{14},\\ &(D_{51} + D_{24})v + D_{61} + D_{34}, 0, D_{71} + D_{17},\\ &(D_{81} + D_{27})v + D_{91} + D_{37}, 0, 0, 0, 0, D_{44},\\ &D_{54}v + D_{64}, 0, D_{74} + D_{47},\\ &(D_{84} + D_{57})v + D_{94} + D_{67}, 0, 0, 0, 0,\\ &D_{77}, D_{87}v + D_{97}, 0, 0, 0, 0),\end{aligned} \tag{70}$$

$$\begin{aligned}\Psi_9 = (&0, 0, 0, D_{12}, D_{22}v + D_{32}, 0, D_{42} + D_{15},\\ &(D_{52} + D_{25})v + D_{62} + D_{35}, 0, D_{72} + D_{18},\\ &(D_{82} + D_{28})v + D_{92} + D_{38}, 0, 0, 0, 0, D_{45},\\ &D_{55}v + D_{65}, 0, D_{75} + D_{48},\\ &(D_{85} + D_{58})v + D_{95} + D_{68}, 0, 0, 0, 0,\\ &D_{78}, D_{88}v + D_{98}, 0, 0, 0, 0),\end{aligned} \tag{71}$$

$$\begin{aligned}\Psi_{10} = (&0, 0, 0, D_{11}, (D_{21} + D_{12})v + D_{13} + D_{31},\\ &D_{22}v^2 + (D_{32} + D_{23})v + D_{33}, D_{41},\\ &(D_{51} + D_{42})v + D_{43} + D_{61},\\ &D_{52}v^2 + (D_{62} + D_{53})v + D_{63}, 0, 0, 0, 0, 0, 0,\\ &D_{71}, (D_{81} + D_{72})v + D_{73} + D_{91},\\ &D_{82}v^2 + (D_{92} + D_{83})v + D_{93},\\ &0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),\end{aligned} \tag{72}$$

$$\begin{aligned}\Psi_{11} = (&0,0,0,D_{14},(D_{24}+D_{15})v+D_{16}+D_{34},\\ &D_{25}v^2+(D_{35}+D_{26})v+D_{36},D_{44},\\ &(D_{54}+D_{45})v+D_{46}+D_{64},\\ &D_{55}v^2+(D_{65}+D_{56})v+D_{66},0,0,0,0,0,0,\\ &D_{74},(D_{84}+D_{75})v+D_{76}+D_{94},\\ &D_{85}v^2+(D_{95}+D_{86})v+D_{96},\\ &0,0,0,0,0,0,0,0,0,0,0,0),\end{aligned} \quad (73)$$

$$\begin{aligned}\Psi_{12} = (&0,0,0,D_{17},(D_{27}+D_{18})v+D_{19}+D_{37},\\ &D_{28}v^2+(D_{38}+D_{29})v+D_{39},D_{47},\\ &(D_{57}+D_{48})v+D_{49}+D_{67},\\ &D_{58}v^2+(D_{68}+D_{59})v+D_{69},0,0,0,0,0,0,\\ &D_{77},(D_{87}+D_{78})v+D_{79}+D_{97},\\ &D_{88}v^2+(D_{98}+D_{89})v+D_{99},\\ &0,0,0,0,0,0,0,0,0,0,0,0),\end{aligned} \quad (74)$$

$$\begin{aligned}\Psi_{13} = (&0,0,0,0,D_{11},D_{21}v+D_{31},0,D_{41}+D_{14},\\ &(D_{51}+D_{24})v+D_{61}+D_{34},0,D_{71}+D_{17},\\ &(D_{81}+D_{27})v+D_{91}+D_{37},0,0,0,0,D_{44},\\ &D_{54}v+D_{64},0,D_{74}+D_{47},\\ &(D_{84}+D_{57})v+D_{94}+D_{67},0,0,0,0,\\ &D_{77},D_{87}v+D_{97},0,0,0),\end{aligned} \quad (75)$$

$$\begin{aligned}\Psi_{14} = (&0,0,0,0,D_{12},D_{22}v+D_{32},0,D_{42}+D_{15},\\ &(D_{52}+D_{25})v+D_{62}+D_{35},0,D_{72}+D_{18},\\ &(D_{82}+D_{28})v+D_{92}+D_{38},0,0,0,0,D_{45},\\ &D_{55}v+D_{65},0,D_{75}+D_{48},\\ &(D_{85}+D_{58})v+D_{95}+D_{68},0,0,0,0,\\ &D_{78},\ D_{88}v+D_{98},0,0,0),\end{aligned} \quad (76)$$

$$\begin{aligned}\Psi_{15} = (&0,0,0,0,0,0,D_{11},(D_{21}+D_{12})v+\\ &D_{13}+D_{31},D_{22}v^2+(D_{32}+D_{23})v+D_{33},D_{41},\\ &(D_{51}+D_{42})v+D_{43}+D_{61},\\ &D_{52}v^2+(D_{62}+D_{53})v+D_{63},0,0,0,0,0,0,\\ &D_{71},(D_{81}+D_{72})v+D_{73}+D_{91},D_{82}v^2+\\ &(D_{92}+D_{83})v+D_{93},0,0,0,0,0,0,0,0,0,0),\end{aligned} \quad (77)$$

$$\begin{aligned}\Psi_{16} = (&0,0,0,0,0,0,D_{14},(D_{24}+D_{15})v+D_{16}+\\ &D_{34},D_{25}v^2+(D_{35}+D_{26})v+D_{36},D_{44},\\ &(D_{54}+D_{45})v+D_{46}+D_{64},D_{55}v^2+\\ &(D_{65}+D_{56})v+D_{66},0,0,0,0,0,0,D_{74},\\ &(D_{84}+D_{75})v+D_{76}+D_{94},D_{85}v^2+\\ &(D_{95}+D_{86})v+D_{96},0,0,0,0,0,0,0,0,0,0),\end{aligned} \quad (78)$$

$$\begin{aligned}\Psi_{17} = (&0,0,0,0,0,0,D_{17},(D_{27}+D_{18})v+\\ &D_{19}+D_{37},D_{28}v^2+(D_{38}+D_{29})v+D_{39},D_{47},\\ &(D_{57}+D_{48})v+D_{49}+D_{67},\\ &D_{58}v^2+(D_{68}+D_{59})v+D_{69},0,0,0,0,0,0,\\ &D_{77},(D_{87}+D_{78})v+D_{79}+D_{97},\\ &D_{88}v^2+(D_{98}+D_{89})v+D_{99},\\ &0,0,0,0,0,0,0,0,0,0),\end{aligned} \quad (79)$$

$$\begin{aligned}\Psi_{18} = (&0,0,0,0,0,0,0,0,0,0,0,0,D_{11},\\ &D_{21}v+D_{31},0,D_{41}+D_{14},\\ &(D_{51}+D_{24})v+D_{61}+D_{34},0,D_{71}+D_{17},\\ &(D_{81}+D_{27})v+D_{91}+D_{37},0,D_{44},\\ &D_{54}v+D_{64},0,D_{74}+D_{47},\\ &(D_{84}+D_{57})v+D_{94}+D_{67},0,D_{77},D_{87}v+D_{97},0),\end{aligned} \quad (80)$$

$$\begin{aligned}\Psi_{19} = (&0,0,0,0,0,0,0,0,0,0,0,0,D_{12},\\ &D_{22}v+D_{32},0,D_{42}+D_{15},\\ &(D_{52}+D_{25})v+D_{62}+D_{35},0,D_{72}+D_{18},\\ &(D_{82}+D_{28})v+D_{92}+D_{38},0,D_{45},D_{55}v+D_{65},\\ &0,D_{75}+D_{48},(D_{85}+D_{58})v+D_{95}+D_{68},\\ &0,D_{78},\ D_{88}v+D_{98},0),\end{aligned} \quad (81)$$

$$\begin{aligned}\Psi_{20} = (&0,0,0,0,0,0,0,0,0,0,0,0,\\ &D_{11},(D_{21}+D_{12})v+D_{13}+D_{31},\\ &D_{22}v^2+(D_{32}+D_{23})v+D_{33},D_{41},\\ &(D_{51}+D_{42})v+D_{43}+D_{61},\\ &D_{52}v^2+(D_{62}+D_{53})v+D_{63},\\ &0,0,0,D_{71},(D_{81}+D_{72})v+D_{73}+D_{91},\\ &D_{82}v^2+(D_{92}+D_{83})v+D_{93},0,0,0,0,0,0),\end{aligned} \quad (82)$$

$$\begin{aligned}\Psi_{21} = (&0,0,0,0,0,0,0,0,0,0,0,0,\\ &D_{14},(D_{24}+D_{15})v+D_{16}+D_{34},\\ &D_{25}v^2+(D_{35}+D_{26})v+D_{36},D_{44},\\ &(D_{54}+D_{45})v+D_{46}+D_{64},\\ &D_{55}v^2+(D_{65}+D_{56})v+D_{66},0,0,0,\\ &D_{74},(D_{84}+D_{75})v+D_{76}+D_{94},\\ &D_{85}v^2+(D_{95}+D_{86})v+D_{96},0,0,0,0,0,0),\end{aligned} \tag{83}$$

$$\begin{aligned}\Psi_{22} = (&0,0,0,0,0,0,0,0,0,0,0,0,\\ &D_{17},(D_{27}+D_{18})v+D_{19}+D_{37},\\ &D_{28}v^2+(D_{38}+D_{29})v+D_{39},D_{47},\\ &(D_{57}+D_{48})v+D_{49}+D_{67},\\ &D_{58}v^2+(D_{68}+D_{59})v+D_{69},0,0,0,\\ &D_{77},(D_{87}+D_{78})v+D_{79}+D_{97},\\ &D_{88}v^2+(D_{98}+D_{89})v+D_{99},0,0,0,0,0,0),\end{aligned} \tag{84}$$

$$\begin{aligned}\Psi_{23} = (&0,0,0,0,0,0,0,0,0,0,0,0,0,0,D_{11},\\ &D_{21}v+D_{31},0,D_{41}+D_{14},\\ &(D_{51}+D_{24})v+D_{61}+D_{34},0,D_{71}+D_{17},\\ &(D_{81}+D_{27})v+D_{91}+D_{37},0,D_{44},D_{54}v+D_{64},\\ &0,D_{74}+D_{47},(D_{84}+D_{57})v+D_{94}+D_{67},\\ &0,D_{77},D_{87}v+D_{97}),\end{aligned} \tag{85}$$

$$\begin{aligned}\Psi_{24} = (&0,0,0,0,0,0,0,0,0,0,0,0,0,0,D_{12},\\ &D_{22}v+D_{32},0,D_{42}+D_{15},\\ &(D_{52}+D_{25})v+D_{62}+D_{35},0,D_{72}+D_{18},\\ &(D_{82}+D_{28})v+D_{92}+D_{38},0,D_{45},D_{55}v+D_{65},\\ &0,D_{75}+D_{48},(D_{85}+D_{58})v+D_{95}+D_{68},\\ &0,D_{78},D_{88}v+D_{98}),\end{aligned} \tag{86}$$

$$\begin{aligned}\Psi_{25} = (&0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\\ &D_{11},(D_{21}+D_{12})v+D_{13}+D_{31},\\ &D_{22}v^2+(D_{32}+D_{23})v+D_{33},\\ &D_{41},(D_{51}+D_{42})v+D_{43}+D_{61},\\ &D_{52}v^2+(D_{62}+D_{53})v+D_{63},0,0,0,\\ &D_{71},(D_{81}+D_{72})v+D_{73}+D_{91},\\ &D_{82}v^2+(D_{92}+D_{83})v+D_{93},0,0,0),\end{aligned} \tag{87}$$

$$\begin{aligned}\Psi_{26} = (&0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\\ &D_{14},(D_{24}+D_{15})v+D_{16}+D_{34},\\ &D_{25}v^2+(D_{35}+D_{26})v+D_{36},\\ &D_{44},(D_{54}+D_{45})v+D_{46}+D_{64},\\ &D_{55}v^2+(D_{65}+D_{56})v+D_{66},0,0,0,\\ &D_{74},(D_{84}+D_{75})v+D_{76}+D_{94},\\ &D_{85}v^2+(D_{95}+D_{86})v+D_{96},0,0,0),\end{aligned} \tag{88}$$

$$\begin{aligned}\Psi_{27} = (&0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\\ &D_{17},(D_{27}+D_{18})v+D_{19}+D_{37},\\ &D_{28}v^2+(D_{38}+D_{29})v+D_{39},\\ &D_{47},(D_{57}+D_{48})v+D_{49}+D_{67},\\ &D_{58}v^2+(D_{68}+D_{59})v+D_{69},0,0,0,\\ &D_{77},(D_{87}+D_{78})v+D_{79}+D_{97},\\ &D_{88}v^2+(D_{98}+D_{89})v+D_{99},0,0,0),\end{aligned} \tag{89}$$

$$\begin{aligned}\Psi_{28} = (&0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\\ &0,0,0,0,0,0,D_{11},(D_{21}+D_{12})v+D_{13}+D_{31},\\ &D_{22}v^2+(D_{32}+D_{23})v+D_{33},\\ &D_{41},(D_{51}+D_{42})v+D_{43}+D_{61},\\ &D_{52}v^2+(D_{62}+D_{53})v+D_{63},\\ &D_{71},(D_{81}+D_{72})v+D_{73}+D_{91},\\ &D_{82}v^2+(D_{92}+D_{83})v+D_{93}),\end{aligned} \tag{90}$$

$$\begin{aligned}\Psi_{29} = (&0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\\ &0,0,0,0,0,0,D_{14},(D_{24}+D_{15})v+D_{16}+D_{34},\\ &D_{25}v^2+(D_{35}+D_{26})v+D_{36},\\ &D_{44},(D_{54}+D_{45})v+D_{46}+D_{64},\\ &D_{55}v^2+(D_{65}+D_{56})v+D_{66},\\ &D_{74},(D_{84}+D_{75})v+D_{76}+D_{94},\\ &D_{85}v^2+(D_{95}+D_{86})v+D_{96}),\end{aligned} \tag{91}$$

$$\begin{aligned}\Psi_{30} = (&0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\\ &0,0,0,0,0,0,D_{17},(D_{27}+D_{18})v+D_{19}+D_{37},\\ &D_{28}v^2+(D_{38}+D_{29})v+D_{39},\\ &D_{47},(D_{57}+D_{48})v+D_{49}+D_{67},\\ &D_{58}v^2+(D_{68}+D_{59})v+D_{69},\\ &D_{77},(D_{87}+D_{78})v+D_{79}+D_{97},\\ &D_{88}v^2+(D_{98}+D_{89})v+D_{99}).\end{aligned} \tag{92}$$

Although the matrix $\mathbf{\Psi}$ is a polynomial in single variable v, it is impractical to compute the determinant of it through determinant expansion directly. To speed up the computation and to conserve memory, an extended row reduction method can be adopted. Let $\boldsymbol{A}$ be a square matrix of polynomials in variable v. Elementary row operations will affect the solutions of the equation Det($\boldsymbol{A}$) = 0 in the following way

a) If $\boldsymbol{A}'$ is the matrix that results from multiplying a row or column of $\boldsymbol{A}$ by a nonzero polynomial c in variable v, then Det($\boldsymbol{A}'$) = cDet($\boldsymbol{A}$). The solutions for the equation Det($\boldsymbol{A}'$) = 0 contain the solutions for the equation Det($\boldsymbol{A}$) = 0.
b) If $\boldsymbol{A}'$ is the matrix that results from interchanging two rows or two columns of $\boldsymbol{A}$ then Det($\boldsymbol{A}'$) = -Det($\boldsymbol{A}$). The solutions for the equation Det($\boldsymbol{A}'$) = 0 are the same as the solutions for the equation Det($\boldsymbol{A}$) = 0.
c) If $\boldsymbol{A}'$ is the matrix that results from adding a multiple of one row of $\boldsymbol{A}$ onto another row of $\boldsymbol{A}$ or adding a multiple of one column of $\boldsymbol{A}$ onto another column of $\boldsymbol{A}$ then Det($\boldsymbol{A}'$) = Det($\boldsymbol{A}$). The solutions for the equation Det($\boldsymbol{A}'$) = 0 are the same as the solutions for the equation Det($\boldsymbol{A}$) = 0.

In traditional computations of the determinants of matrices, one has to be careful with these elementary operations in order to retain the values of the determinants. Since we are only interested in the possible solutions of the equation Det($\boldsymbol{A}$) = 0, we can use these operations at will. For example, from

$$\begin{vmatrix} A_{11} & A_{12} & \vdots & A_{1n} \\ A_{21} & A_{22} & \vdots & A_{2n} \\ \cdots & \cdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix} = 0, \tag{93}$$

We can deduce

$$\begin{vmatrix} A_{11} & A_{12} & \vdots \\ 0 & A_{11}A_{22}-A_{12}A_{21} & \vdots \\ \cdots & \cdots & \vdots \\ 0 & A_{11}A_{n2}-A_{12}A_{n1} & \cdots \end{vmatrix} = 0. \tag{94}$$

A brief examination of the structure of the matrix $\mathbf{\Psi}$ reveals that 10 columns of the matrix contain only constants and 6 columns of the matrix contain some constants. By carefully selecting these constants as pivot values, we can reduce the equation containing a determinant of a 30×30 matrix into an equation containing a determinant of a 14×14 matrix. The transformed determinant is of the form

$$\begin{vmatrix} a_{12}v^2+a_{11}v+a_{10} & b_{12}v^2+b_{11}v+b_{10} & \cdots \\ a_{22}v^2+a_{21}v+a_{20} & b_{22}v^2+b_{21}v+b_{20} & \cdots \\ a_{32}v^2+a_{31}v+a_{30} & b_{32}v^2+b_{31}v+b_{30} & \cdots \\ a_{42}v^2+a_{41}v+a_{40} & b_{42}v^2+b_{41}v+b_{40} & \cdots \\ a_{52}v^2+a_{51}v+a_{50} & b_{52}v^2+b_{51}v+b_{50} & \cdots \\ a_{62}v^2+a_{61}v+a_{60} & b_{62}v^2+b_{61}v+b_{60} & \cdots \\ a_{72}v^2+a_{71}v+a_{70} & b_{72}v^2+b_{71}v+b_{70} & \cdots \\ a_{82}v^2+a_{81}v+a_{80} & b_{82}v^2+b_{81}v+b_{80} & \cdots \\ \vdots & \vdots & \vdots \end{vmatrix}. \tag{95}$$

After the first round of row reductions, the resulting determinant looks like

$$\begin{vmatrix} 0 & \tilde{b}_{14}v^4+\tilde{b}_{13}v^3+\tilde{b}_{12}v^2+\tilde{b}_{11}v+\tilde{b}_{10} & \cdots \\ 0 & \tilde{b}_{23}v^3+\tilde{b}_{22}v^2+\tilde{b}_{21}v+\tilde{b}_{20} & \cdots \\ \tilde{a}_{30} & \tilde{b}_{32}v^2+\tilde{b}_{31}v+\tilde{b}_{30} & \cdots \\ 0 & \tilde{b}_{42}v^2+\tilde{b}_{41}v+\tilde{b}_{40} & \cdots \\ 0 & \tilde{b}_{52}v^2+\tilde{b}_{51}v+\tilde{b}_{50} & \cdots \\ 0 & \tilde{b}_{62}v^2+\tilde{b}_{61}v+\tilde{b}_{60} & \cdots \\ 0 & \tilde{b}_{72}v^2+\tilde{b}_{71}v+\tilde{b}_{70} & \cdots \\ 0 & \tilde{b}_{82}v^2+\tilde{b}_{81}v+\tilde{b}_{80} & \cdots \\ \vdots & \vdots & \vdots \end{vmatrix}. \tag{96}$$

Canceling the constant factor $\tilde{a}_{30}$ and making another round of extended row reductions, we have

$$\begin{vmatrix} 0 & \cdots & \cdots \\ 0 & \cdots & \cdots \\ 0 & \tilde{\tilde{c}}_{44}v^4+\tilde{\tilde{c}}_{43}v^3+\tilde{\tilde{c}}_{42}v^2+\tilde{\tilde{c}}_{41}v+\tilde{\tilde{c}}_{40} & \cdots \\ 0 & \tilde{\tilde{c}}_{53}v^3+\tilde{\tilde{c}}_{52}v^2+\tilde{\tilde{c}}_{51}v+\tilde{\tilde{c}}_{50} & \cdots \\ \tilde{\tilde{b}}_{60} & \tilde{\tilde{c}}_{62}v^2+\tilde{\tilde{c}}_{61}v+\tilde{\tilde{c}}_{60} & \cdots \\ 0 & \tilde{\tilde{c}}_{72}v^2+\tilde{\tilde{c}}_{71}v+\tilde{\tilde{c}}_{70} & \cdots \\ 0 & \tilde{\tilde{c}}_{82}v^2+\tilde{\tilde{c}}_{81}v+\tilde{\tilde{c}}_{80} & \cdots \\ \vdots & \vdots & \vdots \end{vmatrix}. \tag{97}$$

In this way, we can derive a polynomial equation of the form

$$\Pi = \begin{vmatrix} C_{11} & C_{12} & \cdots & C_{1A} \\ C_{21} & C_{22} & \cdots & C_{2A} \\ \vdots & \vdots & \vdots & \vdots \\ C_{A1} & \cdots & \cdots & C_{AA} \end{vmatrix} = 0 , \qquad (98)$$

where **Π** is a determinant of a 10×10 matrix of polynomials in variable v. The elements in the first row of **Π** are polynomials in variable v of the tenth degree, the elements in the second row of **Π** are polynomials in variable v of the ninth degree, the elements in the third row are polynomials in variable v of the eighth degree, etc. The elements in the last two rows are polynomials in variable v of the second degree.

Now performing column reduction on the last three columns of **Π**, we finally obtain a determinant of a 7×7 matrix of polynomials in variable v. Expanding the determinant, we obtain an equation in variable v in explicit form

$$a_n v^n + a_{n-1} v^{n-1} + a_{n-2} v^{n-2} + \cdots + a_1 v + a_0 = 0 . \qquad (99)$$

In summary, the computation steps consist of

a) Compute the coefficient matrix **Φ** in (28) from given point correspondences;
b) Compute cofactors D_{ij} of the coefficient matrix **Φ**;
c) Construct the matrix **Ψ** of polynomials in variable v;
d) Derive equation (99) from **Ψ** through extended row reduction.

Substituting back each solution of v of (99) into (45), we have a polynomial equation in variable u of the sixth degree. For each solution of u and v, substituting them into equations (23), (24), (25), and (26), we can then have a linear solution of f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, and f_{33}. The solution that has the minimum value where the determinant of f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, and f_{33} are of rank 2 is the final result. In this way, we can effectively derive the fundamental matrix with rank-2 constraint.

IV. CONCLUSION

We have presented an algorithm to solve the typical system of multivariable nonlinear equations encountered in the fundamental matrix estimation. The method reduces the system of nonlinear multivariable equations into a single variable equation which is easy to solve. Compared with traditional numerical method based on Newton-Raphson iteration, our method is fast and always produces solutions. Future directions of research include investigating the performance of the algorithm with real data and applying the method to other minimization problems.

REFERENCES

[1] X. Armangué and J. Salvi, "Overall view regarding fundamental matrix estimation," Image and Vision Computing, vol.21, no.2, pp.205–220, February 2003.

[2] A. Bartoli and P. Sturm, "Nonlinear estimation of fundamental matrix with minimal parameters," IEEE Trans. Pattern Analysis and Machine Intelligence, vol.26, no.3, pp.426–432, March 2004.

[3] W. Chojnacki, M.J. Brooks, A. van den Hengel and D. Gawley, "A new constrained parameter estimator for computer vision applications," Image and Vision Computing, vol.22, no.2, pp.85–91, February 2004.

[4] O. Faugeras, "What can be seen in three dimensions with an uncalibrated stereo rig?" Proc. 2nd European Conference on Computer Vision, pp.563–578, Santa Margherita Ligure, 1992

[5] R. Hartley, "Estimation of relative camera positions for uncalibrated cameras," Proc. 2nd European Conference on Computer Vision, pp.579–587, Santa Margherita Ligure, 1992.

[6] R.I. Hartley, "In defense of the eight-point algorithm," IEEE Trans. Pattern Analysis and Machine Intelligence, vol.19, no.6, pp.580–593, June 1997

[7] R. Hartley and A. Zisserman, Multiple View Geometry in Computer Vision. Cambridge University Press, Cambridge, U.K., 2000.

[8] P.J. Huber, "Robust statistics," Wiley, New York, 1981.

[9] K. Kanatani and Y. Sugaya, "High accuracy fundamental matrix computation and its performance evaluation," IEICE Trans. Information and Systems, vol.E90-D, no.2, pp.579–585, February 2007.

[10] K. Kanatani and Y. Sugaya, "Extended FNS for constrained parameter estimation," Proc. 10th Meeting Image Recognition and Understanding, pp.219–226, July 2007.

[11] H.C. Longuet-Higgins, "A computer algorithm for reconstructing a scene from two projections," Nature, vol.293, no.10, pp.133–135, September 1981.

[12] Q.-T. Luong and O. Faugeras, "The fundamental matrix: theory, algorithms, and stability analysis," The International Journal of Computer Vision, vol.17, no.1, pp.43-76, January 1996.

[13] T. Migita and T. Shakunaga, "One-dimensional search for reliable epipole estimation," Proc. IEEE Pacific Rim Symp. Image and Video Technology, pp.1215–1224, December 2006.

[14] Y. Sugaya and K. Kanatani, "High accuracy computation of rank-constrained fundamental matrix," Proc. 18th British Machine Vision Conf., vol.1, pp.282–291, September 2007.

[15] Z. Zhang, R. Deriche, O. Faugeras and Q.-T. Luong, "A robust technique for matching two uncalibrated images through the recovery of the unknown epipolar geometry," Artificial Intelligence, vol.78, pp.87–119, 1995.

[16] Z. Zhang, "Determining the epipolar geometry and its uncertainty: a review," International Journal of Computer Vision, vol.27, no.2, pp.161–195, March 1998.

[17] Z. Zhang and C. Loop, "Estimating the fundamental matrix by transforming image points in projective space," Computer Vision and Image Understanding, vol.82, no.2, pp.174–180, May 2001.