Structural Controllability of Switching Linear Systems

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Abstract— **This paper investigates the structural controllability problem for controlled switching linear systems. Causal manipulations on the bond graph models are carried out in order to determine graphically the controllable state subspaces. Graphical conditions for structural controllability are derived by using these controllable state subspaces.**

Index Terms— **Hybrid systems, Switching systems, Bond graph, Structural controllability.**

I. INTRODUCTION

A broad class of hybrid systems is composed of physical processes with switching devices. Such processes are called switching systems and are very common in various engineering fields (e.g. hydraulic systems with valves,.., electric systems with diodes, relays,…, mechanical systems with clutches...). These systems are characterized by a Finite State Automaton (FSA) and a set of dynamic systems, each one corresponding to a state of the FSA. The change of states can be either controlled or autonomous. Various researchers investigated this problem using the bond graph tool [1]-[4].

Several concepts appeared in the last decade addressing the controllability problem: controllable sublanguage concept [6], hybrid controllability concept [7], between-block controllability concept [8]. Controlled switching linear systems (CSLS) on which we focus in this work belong to the hybrid controllability concept as they address a reachability problem of hybrid states. The characteristics of CSLS are: all mode switches are controllable, the dynamical subsystem within each mode has a linear time invariant form, the admissible region of operation within each mode is the whole state and input space, and there are no discontinuous state jumps.

On the other hand, the bond graph concept, introduced by H. M. Paynter, is an alternate representation of physical systems. Some recent works have enabled to highlight structural properties of these systems [9]. In [4]-[9], the structural controllability/ observability property is studied with the aid of simple causal manipulations on the bond graph model.

The aim of this work is to investigate the structural controllability for controlled switching linear systems modelled by bond graph. This paper is organized as follows: the second section formulates the CSLS controllability. Section three recalls some background about bond graph modelling of hybrid systems with ideal switches. In section four the structural controllability of these systems is studied using algebraic characterization. Graphical conditions and procedures are then proposed. Finally, a simple example illustrating the previous results is proposed.

II. CONTROLLABILTY OF CONTROLLED SWITCHING LINEAR SYSTEMS

Consider a Controlled Switching Linear Systems [5], given by equation (1):

$$\dot{x}(t) = A(\sigma(t))x + B(\sigma(t))u \tag{1}$$

Where $x \in R^n$ is the state variable, $u \in R^m$ is the input variable, σ is a piecewise constant switching function and (σ_i, x) the hybrid state.

If we consider this system in a particular mode *i*, the equation (1) can be written as:

$$\dot{x} = A_i x + B_i u \tag{2}$$

With, $A_i = A(\sigma_i)$, $B_i = B(\sigma_i)$, $i \in \{1, \cdots, q\}$ and q the number of mode.

Remark 1: System (2) can be considered as a linear time invariant system (LTI).

Assumptions

1) We suppose that A_i and B_i matrices are constant on a time interval $[t_0, t_0 + \tau)$, where $\tau \geq \tau_{\min} > 0$, and the constant $\tau_{\min}$ is arbitrarily small and independent of mode i. For instance, we suppose that the dynamics in (1) are given by $\dot{x} = A_i x + B_i u$ over the finite time interval $[t_k, t_{k+1})$. At time t_{k+1}, the dynamics in interval $[t_{k+1}, t_{k+2})$ are given by $\dot{x} = A_j x + B_j u$.

2) We assume that the state vector $x(t)$ does not jump discontinuously at t_{k+1}.

Definition 1 [5] Given any pair of hybrid states, (σ_0, x_0) and (σ_q, x_q), if there exists a timed mode-switching set $\{(\sigma_{i-1}, t_i, \sigma_i)\}_{i=1}^{q}$ and a corresponding piecewise continuous-finite input signal $u(t)$, such that

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system (1) is reachable from (σ_0, x_0) to (σ_q, x_q) within a finite time interval, then the considered system (1) is controllable, otherwise, system (1) is uncontrollable.

A. An algebraic sufficient condition

[5, Yang] proposed a sufficient condition of CSLC controllability using combined matrix.

The controllability combined matrix W_C of system (1) is given by equation 3:

$$W_C \triangleq [W_1 W_2 \cdots W_q] \tag{3}$$

Where $W_i \triangleq [B_i \ A_i B_i \cdots A_i^{n-1} B_i]$ is the controllability matrix of system (2).

Theorem 1 [5] The CSLS (1) with q modes, is controllable, if the controllability matrix W_C defined in (3) is of full row rank.

Remark 2: From this theorem, we can deduce that:
- The system (1) can be controllable, if there is only one controllable system (2).
- However, it is possible that no system (2) is controllable but that the system (1) is controllable.

B. A necessary and sufficient algebraic condition

Using the joint controllability matrices, [5] proposed a necessary and sufficient algebraic condition.

Let us define the (n , mn^k) matrix (equation 4):

$$Œ^k(i, i_1, \cdots, i_k) \triangleq [A_{i_k}^{j_k} \cdots A_{i_2}^{j_2} A_{i_1}^{j_1} A_i^j B_i]_{j_k, \cdots, j_1, j \in \{0,1,\cdots,n-1\}} \tag{4}$$

Based on the definition of Œ we construct a new matrix Æ as follows:

$$\begin{cases} Æ^0(i) \triangleq Œ^1(i) = W_i \\ Æ^1(i) \triangleq [Œ^2(i, j)]_{j \in \{1,\cdots,q\} \text{ and } i \neq j} \\ \vdots \\ Æ^k(i) \triangleq [Œ^{k+1}(i, i_1, \cdots, i_k)]_{i_1, \cdots, i_k \in \{1,\cdots,q\}} \end{cases} \tag{5}$$

With $i_1 \neq i, \cdots, i_k \neq i_{k-1}$

The joint controllability matrices can be defined as :

$$\begin{cases} \overline{W}^0 = [Æ^0(1) \cdots Æ^0(q)] \\ \vdots \\ \overline{W}^k = [Æ^k(1) \cdots Æ^k(q)] \end{cases} \tag{6}$$

$\overline{W}^k$ is the kth-order joint controllability matrix of the system (1). There exists a joint controllability coefficient k_r of the system, which is defined as [5]:

$$k_r \triangleq \arg \min_l (\text{rank}(\overline{W}^l) = \text{rank}(\overline{W}^{l+1})) \tag{7}$$

Theorem 2 [5] System (1) is controllable, if and only if $\text{rank}(\overline{W}^{k_r}) = n$.

III. BOND GRAPH APPROACH

The structure junction of a switching bond graph can be represented by figure 1. Five fields model the components behaviour, 4 that belong to the standard bond graph formalism; - source field which produces energy, - *R* field which dissipates it, - *I* and *C* field which can store it, and the *Sw* field that is added for switching components. This element is made of the power variables imposed by the switches in the chosen configuration.

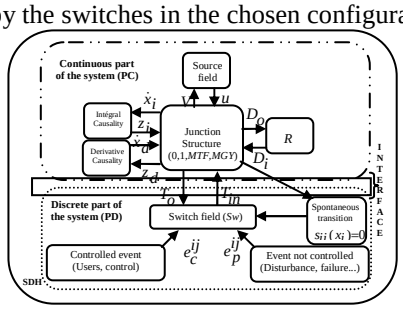

Figure 1. Structure of junction

In order to model and analyze hybrid systems in the BG framework, additional elements are necessary to capture the discrete switching and the change in the model configuration. These elements are called switch elements and can be either controlled or autonomous. An automaton can be used to model the discrete control and the autonomous changes of switches. The location of the automaton defines the switch configurations. The continuous part of hybrid physical systems can be modelled using two main bond graph approaches: non ideal switches with constant circuit topology [1] or ideal switches with variable circuit topology [2].

Assumption

To take into account the absence of discontinuities, we suppose that there are no elements in derivative causality in the bond graph model in integral causality, before and after commutation;

Using the structure junction, the following equation is given [9]:

$$\begin{pmatrix} \dot{x} \\ D_0 \\ T_{0_i} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{13} & S_{14} & S_{15} \\ -S_{13}^T & S_{33} & S_{34} & S_{35} \\ -S_{14}^T & -S_{34}^T & S_{44} & S_{45} \end{pmatrix} \begin{pmatrix} z \\ D_i \\ T_{in_i} \\ u \end{pmatrix} \tag{8}$$

$D_i = LD_0$, L is a positive matrix. Let assume that $H = L(I - S_{33}L)^{-1}$ is an invertible positive matrix. Then the second row leads to $D_i = -HS_{13}^T Fx + HS_{35}u + HS_{34}T_{in_i}$

The third line of (8) gives:

$$T_{0_i} = (-S_{14}^T + S_{34}^T HS_{13}^T)Fx + (S_{45} - S_{34}^T HS_{35})u + (S_{44} - S_{34}^T HS_{34})T_{in_i}$$

The substitution in the first line of (8) gives:

$$\dot{x} = A_i x + B_{ci} u + B_{di} T_{in_i} \tag{9}$$

This system is equivalent to system (2), where $A_i = (S_{11} - S_{13} H S_{13}^T)F$, $B_{ci} = S_{15} + S_{13} H S_{35}$ and $B_{di} = S_{14} + S_{13} H S_{34}$.

Therefore, for N switchs, we have $2^N = q$ modes:

$$\begin{cases} \dot{x}(t) = A_1 x(t) + B_{c1} u(t) + B_{d1} T_{in_1}(t) & t \in [t_0, t_1) \\ \vdots & \vdots \\ \dot{x}(t) = A_q x(t) + B_{cq} u(t) + B_{dq} T_{in_q}(t) & t \in [t_{q-1}, t_q) \end{cases} \quad (10)$$

This system is equivalent to system (1).

IV. STRUCTURAL CONTROLLABILITY

The bond graph concept is an alternate representation of physical systems. Many works allow to highlight structural properties of these systems [3]-[4]. In [4], the structural controllability property is studied using simple causal manipulations on the bond graph model. It is shown that the structural rank concept is somewhat different for bond graph models because it is more precise than for other representations. Our objective is to extend those properties to CSLS systems.

In the following, BGI and BGD denote respectively the bond graph model when the preferential integral (respectively derivative) causality is affected.

To study structural controllability of CSLS modelled by bond graph three graphical methods are proposed: two sufficient conditions and a necessary and sufficient one. Formal representation of controllability subspace is given for bond graph models. It is calculated through causal manipulations. The base of this subspace is used to propose a procedure to study the system controllability.

A. Graphical sufficient condition 1

A system (1) with q modes is controllable if only one system (2) is controllable. This condition can be interpreted by using the result of structural controllability of LTI system.

Indeed, this result is a simple recovery of those giving the necessary and sufficient condition of structural controllability of LTI system modelled by bond graph approach [4].

Theorem 3 The CSLS system (10) is structurally state controllable if:

- All dynamic elements in integral causality are causally connected with a discrete or a continuous input control,
- BG-rank$[A_i \ \mathrm{B}_i] = n$, with $\mathrm{B}_i = [B_{ci} \ B_{di}]$, $i \in \{1, \cdots, q\}$.

Property 1:

BG-rank $[A_i \ \mathrm{B}_i]$ = rank$(S_{11} \ S_{13} \ S_{14} \ S_{15}) = n - t^i_{Sw_s}$.

Where $t^i_{Sw_s}$ is the number of elements remaining in integral causality in BGD$_i$, when a dualism of the maximum number of continuous and discrete input sources is applied (in order to eliminate elements in integral causalities).

Example 1. We consider the following acausal bond graph model:

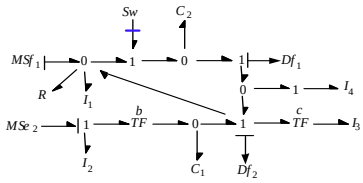

Figure 2: Acausal bond graph model

There are six state variables P_i on I_i, q_j on C_j ($i = 1, \cdots 4$; $j = 1,2$). The dimension of the system is $n = 6$. For models BGI_1 and BGI_2 all state variables are causally connected with the sources, and are in integral causality. We have one switch, then the number of possible configurations is $2^1 = 2$. There is no storing element in derivative causality in this configuration, so the implicit and explicit state variables are the same and are given by: $x = \begin{pmatrix} P_{I_1} & P_{I_2} & P_{I_3} & P_{I_4} & q_{C_1} & q_{C_2} \end{pmatrix}^t$. The bond graph models in integral causality for these two modes are given by figure 3.

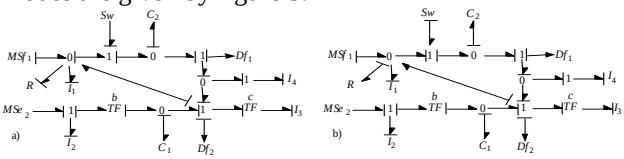

Figure 3: a) BGI$_1$: Sw : $Se = 0$, b) BGI$_2$: Sw : $Sf = 0$

The application of the derivative causality, for example on mode 1 (figure 3.b), give the following BGD$_1$ (figure 4).

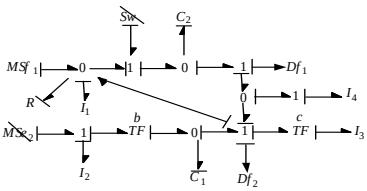

Figure 4: Application of theorem 3

We have BG-rank$[A_i \ \mathrm{B}_i] = 6$, this mode is controllable by continuous input MSf_1 and MSe_2 and discrete input T_{in_i}, then the system is structurally controllable.

To study the controllability of system (10), it is necessary to apply this result to all modes; if one controllable mode exists, the procedure is stopped.

The case where no mode is controllable, but when the system is controllable, can be verified by formal calculation of combined matrix (3). This calculation can be formally effected by using the bond graph model in integral causality [3], or by calculating the controllability subspace from bond graph model in derivative causality. We chose to translate the latter in the form of a second sufficient condition.

C. Graphical sufficient condition 2

From the BGD$_i$, t^i_s algebraic equations can be written (equation 13):

$$g_k^i - \sum_r \alpha_r^{ik} g_r^i = 0 \tag{13}$$

- t_s^i is the number of elements in integral causality in BGD_i, when a dualization of the maximum number of continuous input sources is applied (in order to eliminate elements in integral causalities);
- g_k^i is either an effort variable e_r for I-element in integral causality or a flow variable f_r for C-element in integral causality;
- g_r^i is either an effort variable e_r for I-element in derivative causality or a flow variable f_r for C- element in derivative causality;
- α_r^{ik} is the gain of the causal path between the kth I or C-elements in integral causality and the rth I or C-elements in derivative causality.

Let us consider the t_s^i row vectors z_k^i $(k = 1, \cdots, t_s^i)$ whose components are the coefficients of the variables g_l^i $(l = k, r)$ in the equation (13).

Property 2 [9] The t_s^i row vectors z_k^i $(k = 1, \cdots, t_s^i)$ are orthogonal to the structural controllability subspace vectors of the ith mode. We write $Z_i = (z_k^i)_{k=1,\cdots,t_s^i}$ and $R_0^{i\perp} = \mathrm{Im}(Z_i)$.

Using the bond graph model in derivative causality, the uncontrollable $R_0^{i\perp}$ subspace can be calculated:

Procedure 1: Calculation of $R_0^{i\perp}$

1) On the BGD_i, dualize the maximum number of input sources in order to eliminate (if it is possible) the elements remaining in integral causality,
2) For each element remaining in integral causality, write the algebraic relations with elements in derivative causality (equation 13),
3) Write a row vector z_k^i for each algebraic relation with the causal path gains and write $Z_i = (z_k^i)_{k=1,\cdots,t_s^i}$.

Example 2. The derivative causality and dualization are now applied to the previous bond graph model where the second effort source is removed. The corresponding bond graph models are drawn on figure 5.

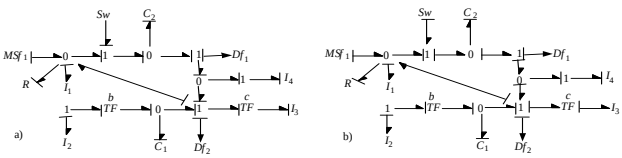

Figure 5 Bond graph models in derivative causality a) BGD_1, b) BGD_2

For mode 1, the dynamic elements I_3 and I_4 has in integral causality. Thus $t_s^1 = 2$, this mode is not structurally controllable. We can write $e_{I_1} - e_{I_4} = 0$, $\frac{1}{b} e_{I_2} + e_{I_3} = 0$, the dimension of the controllability subspace for this mode is 4, because we have four dynamic elements in derivative causality. The basis of the orthogonal to this structural controllability subspace is given by these two vectors $z_1^1 = (1 \quad 0 \quad 0 \quad -1 \quad 0 \quad 0)$ and $z_2^1 = (0 \quad \frac{1}{b} \quad c \quad 0 \quad 0 \quad 0)$.

For mode 2, the dynamic element I_4 is in integral causality, so we can write $e_{I_1} + \frac{1}{b} e_{I_2} + c e_{I_3} - e_{I_4} = 0$, thus we obtain $z_1^2 = (1 \quad \frac{1}{b} \quad c \quad -1 \quad 0 \quad 0)$.

In order to calculate an R_0^i basis, it is enough to find $n - t_s^i$ (n is the system order) independent column vectors w^{ir} $(r = 1, \cdots, n - t_s^i)$. These vectors are gathered in the matrix $W^i = (w^{ir})_{r=1,\cdots,n-t_s^i}$.

From the BGD_i (and dualization of inputs sources), $n - t_s^i$ algebraic relations can be written (14).

$$g_r^i - \sum_k \gamma_k^{ir} g_k^i = 0 \tag{14}$$

- g_r^i is either a flow variable f_r for I-element in derivative causality or an effort variable e_r for C-element in derivative causality;
- g_k^i is either a flow variable f_r for I-element in integral causality or an effort variable e_r for C- element in integral causality;
- γ_k^{ir} is the gain of the causal path between the rth element in derivative causality and the kth element in integral causality.

Suppose now $n - t_s^i$ column vectors w^{ir} whose components are the coefficients of g_r^i and g_k^i variables in equation (14).

Procedure 2 : Calculation of R_0^i

1) On the BGD_i, dualize the maximum number of continuous input sources in order to eliminate (if possible) the elements in integral causality;
2) For each element in derivative causality, write the algebraic relations with elements in integral causality (equation 14);
3) Write a column vector w^{ir} for each algebraic relation with the causal path gains (equation 14), with $R_0^i = \mathrm{Im}(W^i)$.

Property 3: $n - t_s^i$ column vectors w^{ir} $(r = 1, \cdots, n - t_s^i)$ compose a basis for the structural controllability subspace of ith mode.

Example 3. We implement the procedure 2 on the previous example. For mode 1, the two dynamic elements

C_1 and C_2 are not causally connected with I_3 and I_4, we can write $e_{c_1} = e_{c_2} = 0$, the two corresponding vectors are $w^{13} = (0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0)^t$ and $w^{14} = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1)^t$. The algebraic equations corresponding to the elements I_1 and I_2 are given by:

$bf_{I_2} - \frac{1}{c} f_{I_3} = 0 \quad \Rightarrow \quad w^{11} = (0 \quad b \quad -\frac{1}{c} \quad 0 \quad 0 \quad 0)^t$ and

$f_{I_1} + f_{I_4} = 0 \quad \Rightarrow w^{12} = (1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0)^t$ and $R_0^1 = \mathrm{Im}(\{w^{11}, w^{12}, w^{13}, w^{14}\}$.

In mode 2, we have $e_{c_1} = e_{c_2} = 0$. The two corresponding vectors are $w^{24} = (0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0)^t$ and $w^{25} = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1)^t$. The algebraic equations corresponding to elements I_1, I_2 and I_3 are given by:

$bf_{I_2} + f_{I_4} = 0$, $\frac{1}{c} f_{I_3} + f_{I_4} = 0$ and $f_{I_1} + f_{I_4} = 0$; Then

$w^{21} = (0 \quad b \quad 0 \quad 1 \quad 0 \quad 0)^t$, $w^{22} = (0 \quad 0 \quad \frac{1}{c} \quad 1 \quad 0 \quad 0)^t$,

$w^{23} = (1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0)^t$ and $R_0^2 = \mathrm{Im}\{w^{21}, w^{22}, w^{23}, w^{24}, w^{25}\}$

The graphical calculation of structural controllability subspaces and theorem 1 leads to proposition 3:

Proposition 3 [9] If $\mathrm{rank}[W^1 \cdots W^q] = n$, the system CSLS (9) is structurally controllable.

Proof: We have shown for a given mode that the bond graph model in derivative causality is characterized by an algebraic equation of the form (14) from which we construct a W^i base of structural controllability subspace of the ith mode, denoted $R_0^i = \mathrm{Im}\{W^i\}$, with $W^i = (w^{ir})_{r=1,\cdots,n-t_s^i}$.

After commutation from i^{th} mode to $(i+1)^{th}$ mode, implement a derivative causality on the bond graph model and dualization the maximum number of input sources, We can write another algebraic relation (equation 15).

$$g_r^{i+1} - \sum_k \gamma_k^{(i+1)r} g_k^{i+1} = 0 \tag{15}$$

Its base is given by $W^{i+1} = (w^{(i+1)k_{i+1}})_{k_{i+1}=1,\cdots,n-t_s^i}$.

However, the condition: $\mathrm{rank}[W_1 \cdots W_q] = n$ is sufficient for system controllability, which implies that the condition $\mathrm{rank}[W^1 \cdots W^i W^{i+1} \cdots W^q] = n$ is also sufficient.■

Example 4. Proposition 3 is now applied to the previous bond graph model, we have $Rank\left(w^{11} \; w^{12} \; w^{13} \; w^{14} \; w^{21} \; w^{22} \; w^{23} \; w^{24} \; w^{25}\right) = 6$, then, this system is structurally controllable.

If this proposition is not verified, then it is necessary to have a necessary and sufficient condition.

D. Graphical necessary and sufficient condition

The CSLS system is given by (1) or (10) without switching digraph, i.e., there is no restriction on the design of switching signal for system $\sum$.

From the BGD_i (and dualization of inputs sources), the following relation can be calculated for each switch:

$$T_{o_i} - g'^i_r - \sum_k \gamma_k^{ir} g'^i_k = 0 \tag{16}$$

- T_{o_i} is the variable on the switch, outgoing of the junction structure; [2]-[9].
- g'^i_r and g'^i_k can be effort or flow;
- γ_k^{ir} is defined in equation 14.

From (14) we propose the invariants for the model in derivative causality:

Proposition 4 [9] The invariant associated to each switch for model BGD_i associated to system (10) is given by the inequality constraints:

$$Inv^d(\sigma_i): \; T_{o_i} = g'^i_r + \sum_k \gamma_k^{ir} g'^i_k > 0 \tag{17}$$

At the commutation, from equation 17 and after cancelling T_{o_i}, N conditions can be given:

$$g'^i_r + \sum_k \gamma_k^{ir} g'^i_k = 0 \tag{18}$$

g'^i_r, g'^i_k and γ_k^{ir} are defined in equation 13.

Suppose now the $t_{Sw_s}^{(i-1)\to i}$ column vectors $w_{Sw_{js}}^{(i-1)\to i}$ $(j = 1, \ldots, N)$ whose components are the coefficients of the variables g'^i_r and g'^i_k in equation (18).

$t_{Sw_s}^{(i-1)\to i}$ is the number of inequalities constraints, from which we have these column vectors $w_{Sw_{js}}^{(i-1)\to i}$ $(j = 1, \ldots, N)$.

In the next step, and for case where no mode is controllable, we propose a method to calculate the total subspace.

Procedure 3 : Calculation of R_0

1) On the BGD_i, dualize the maximum number of continuous input sources in order to eliminate (if it is possible) the elements in integral causality;
2) write the relation between each switch element and the dynamic elements;
3) Deduce the $t_{Sw_s}^{(i-1)\to i}$ invariants for the corresponding BGD_i;
4) Write the conditions of commutation using (18);
5) Write a column vector $w_{Sw_{js}}^{(i-1)\to i}$ ($j = 1, \cdots, N$) for each algebraic relation with the causal path gains;
6) Check if $z_k^i . w_{Sw_{js}}^{(i-1)\to i} = 0$, and write $R_0 = \mathrm{Im}(W^{1k_1} \; W^{1\to 2} \cdots W^{q-1\to q})$, With

$$W^{i-1\to i} = [W^{ik_i}\ w_{Sw_{js}}^{(i-1)\to i}]_{k_i=1,\cdots,n-t_s^i,\ i=1,\cdots,q};$$

Proposition 5 [9] System (10) is structurally controllable, if and only if $\mathrm{rank}(W^{1k_1}\ W^{1\to 2}\cdots W^{q-1\to q}) = n$.

The following procedure summarises the steps to be followed to study the controllability of a CSLS modelled by bond graph.

Procedure 4

1) For each mode $i \in \{1,\cdots,q\}$; if all elements in integral causality are causally connected with an discrete or a continuous control, and if BG-rank$[A_i\ \mathrm{B}_i] = n$, then the system (9) is controllable (sufficient condition 1), if not;

2) Calculate $R_o^i = \mathrm{Im}(W^{ik_i})$, the large controllable subspace (chosen mode);

3) For one mode (except for the already chosen one), calculate the subspace containing the incontrollable variables in the chosen mode;

4) Repeat step 3 for the other modes,

If $\mathrm{rank}[W^1 W^2 \cdots W^q] = n$, the system CSLS (10) is controllable (sufficient condition 2), If not

5) If $\mathrm{rank}(W^{1k_1}\ W^{1\to 2}\cdots W^{q-1\to q}) = n$, then the system (10) is controllable, if not, the system is not controllable.

V. APPLICATION

Consider the mechanical sketch of figure. 6. Two ideal mechanical couplers indicated $Sw_{1,2}$ can be noticed, which serve to couple the mass+two-dampers system in the middle to the mass-spring-damper systems on the sides. The following switching conditions are assumed for each coupler: *i*) switch closes on contact; *ii*) switch opens when $b_{1/2}$-damper compression force becomes zero. We identify the switch binary states as follows: Sw_i=0, switch open (disengaged); Sw_i=1, switch closed (engaged). The SwBG consists then of four switching modes, corresponding to the four binary states of the pair (Sw_1,Sw_2) {(0,0), (1,0), (0,1), (1,1)}.

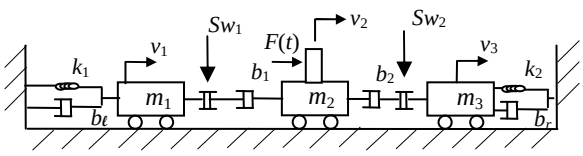

Figure 6. Mechanical system

Figure 9 is a bond graph representation of the example system using the switch element. Naturally, here all the BG-elements are present, even if they may not be active in some modes. Like in the previous series of BGs, causality has also been indicated here: note that the causal stroke is pictured in the middle of a bond when this changes causality in dependence of the switch state, i.e., causality is undecided unless the switch state is specified. It means that the switched system is only specified up-to causality when using this formalism to construct the bond graph model.

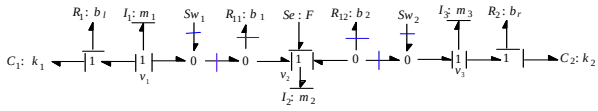

Figure 7. The bond graph model in integral causality.

The four bond graph models in integral causalities BGI_1, BGI_2, BGI_3 and BGI_4 are associated respectively to mode 1, mode 2, mode 3 and mode 4 (figure. 8).

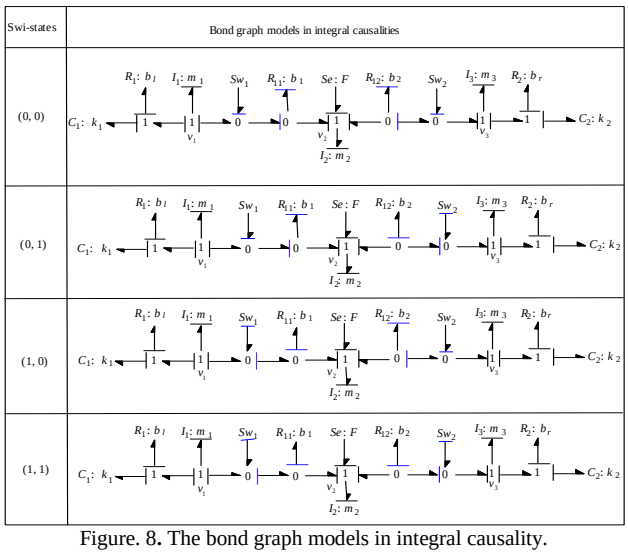

Figure. 8. The bond graph models in integral causality.

The bond graph models in derivative causality of these modes are given in figure. 9:

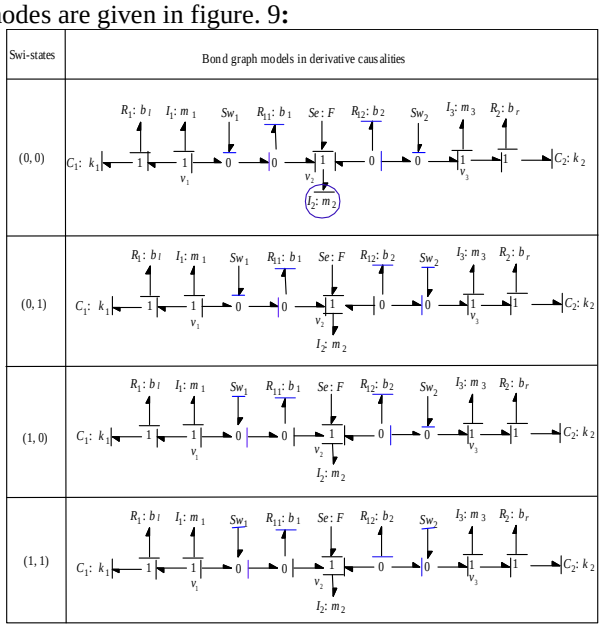

Figure 9. The bond graph models in derivative causality.

This system is controllable because we have at least one controllable mode.

Procedure 4 is now applied on the following acausal (without causality) bond graph model (figure 10):

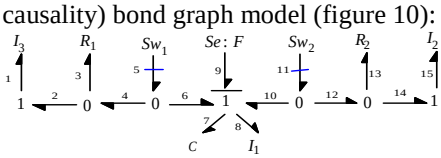

Figure 10. Acausal bond graph model

This bond graph contains two switches (Sw_1 and Sw_2), so 4 modes are possible: Mode 1 (Sw_1 closed, Sw_2 open), Mode 2 (Sw_1 open, Sw_2 closed), Mode 3 (Sw_1 open, Sw_2 open). But only three are considered (Mode 4 (Sw_1 closed, Sw_2 closed) is not practicable).

The three bond graph models in integral causalities BGI_1, BGI_2 and BGI_3 are associated respectively to mode 1, mode 2 and mode 3 (figures 11, 12 and 13).

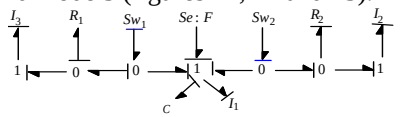

Figure 11. BGI_1: BG of mode 1 in integral causality

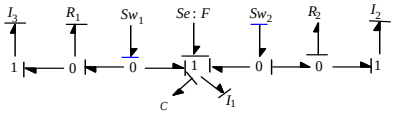

Figure 12. BGI_2: BG of mode 2 in integral causality

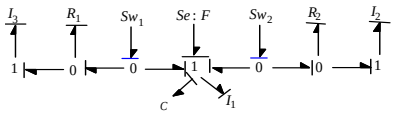

Figure 13. BGI_3: BG of mode 3 in integral causality

Application of procedure 4

Step 1: - This step is applied to the 3 modes, but only the mode 1 is presented.

- On the BGI_1, all the elements in integral causality are connected to a switch (Sw_1 or Sw_2),
- On the BGD_1, one element stays in integral causality (figure 14), and the dualization of MSe does not change its causality;
- So this mode is not controllable.

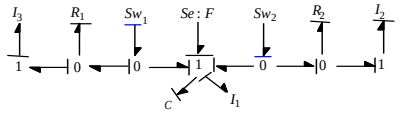

Figure 14. BGD_1: BG model in derivative causality of mode 1

In the same way, the other two other modes are not controllable, therefore, step 1 is not verified.

Step 2: The BGD of modes 1, 2 and 3 contains respectively 3, 3 and 2 elements in derivative causality; therefore we can start with mode 1 or mode 2. We will use mode1.

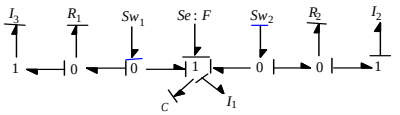

Figure 15. BGD_2: BG model in derivative causality of mode 2

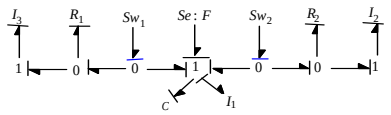

Figure 16. BGD_3: BG model in derivative causality of mode 3

In the BGD_1 (figure 14), I_2 remain in integral causality, we can write $e_{I_2} = 0$, thus $z_1^1 = (0100)$. The dynamic elements I_1, I_3 and C, are not causally connected with I_2. So $f_{I_1} = f_{I_3} = 0$ $e_c = 0$. The three corresponding vectors are $w^{11} = (1000)^t$, $w^{12} = (0010)^t$ and $w^{13} = (0001)^t$. We have $Z_1.W^1 = 0$, then $R_0^1 = \mathrm{Im}(w^{11}, w^{12}, w^{13})$, with $\mathrm{rank}(W^1) = 3$.

Step 3: - In the BGD_2 (figure 15), I_3 remain in integral causality, we write: $e_{I_3} = 0$ therefore $z_1^2 = (0010)$. I_1, I_2 and C, are not causally connected with I_3. So $f_{I_1} = f_{I_2} = 0$, $e_c = 0$, $w^{21} = (1000)^t$, $w^{22} = (0100)^t$ and $w^{23} = (0001)^t$. We have $Z_2.W^2 = 0$ then $R_0^2 = \mathrm{Im}(w^{21}, w^{22}, w^{23})$ with $\mathrm{rank}(W^2) = 3$.

- In the BGD_3, I_2 and I_3 remain in integral causality. Thus $e_{I_2} = 0$ $e_{I_3} = 0$, $z_1^3 = (0100)$, and $z_2^3 = (0010)$. I_1 and C, are not causally connected with I_2 and I_3. So $f_{I_1} = 0$ $e_c = 0$, $w^{31} = (1000)^t$ and $w^{32} = (0001)^t$. We have $Z_3.W^3 = 0$ then $R_0^3 = \mathrm{Im}(w^{31} w^{32})$ with $\mathrm{rank}(W^3) = 2$.

From procedure 4 (step 4), we have:

$$rank \begin{bmatrix} \langle 1 \rangle & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \langle 1 \rangle & 0 \\ 0 & \langle 1 \rangle & 0 & 0 & 0 & 0 \\ 0 & 0 & \langle 1 \rangle & 0 & 0 & 1 \end{bmatrix} = 4 .$$

Then the system is controllable.

Remark 3. This result can also be verified by applying step 4:

Calculation of R_0.

The invariants ($Inv^D(\sigma_1), Inv^D(\sigma_2)$ and $Inv^D(\sigma_3)$) can be computed according to equation (17):

Mode 1: $e_{Sw_1} = e_{I_3}$, $f_{Sw_2} = f_{I_1} + f_{I_2}$, mode 2: $e_{Sw_2} = e_{I_2}$, $f_{Sw_1} = f_{I_1} + f_{I_3}$ and mode 3: $f_{Sw_1} = f_{I_1} + f_{I_3}$, $f_{Sw_2} = f_{I_1} + f_{I_2}$.

We suppose that mode 1 is the initial mode, therefore it is characterized by its controllable subspace $R_0^1 = \mathrm{Im}(w^{11}, w^{12}, w^{13})$ and its inequality constraints $e_{Sw_1} = e_{I_3} > 0$, $f_{Sw_2} = f_{I_1} + f_{I_2} > 0$, and the timed mode-switching set is $\sigma_1 \to \sigma_2 \to \sigma_3$.

After commutation (for example commutation towards mode 2 : [1→2]), we have on the one hand:

- $z_1^2 = (0010)$, $w^{21} = (1000)^t$, $w^{22} = (0100)^t$ and $w^{23} = (0001)^t$; of another share, we have: $w_{Sw_2}^{1\to 2} = (1100)^t$, because $z_1^2 w_{Sw_2}^{1\to 2} = 0$, thus

$$R_0^{1\to 2} = \mathrm{Im}(w^{21}, w^{22}, w^{23}, w_{Sw_2}^{1\to 2}) = \mathrm{Im}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right\},$$

[2→3]: $R_0^{2\to 3} = \mathrm{Im}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\},$

[3→1]:

$$R_0^{3\to 1} = \mathrm{Im}(w^{11}, w^{12}, w^{13}, w_{Sw_1}^{3\to 1}) = \mathrm{Im}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}\right\},$$

The application of procedure 3 gives $R_0 = \mathrm{Im}(W^{1k_1} \quad w^{1\to 2} w^{2\to 3} w^{3\to 1}) = R^4$.

Remark 4. This calculus can also be used to construct the hybrid automaton (figure 17) witch is not used in this work

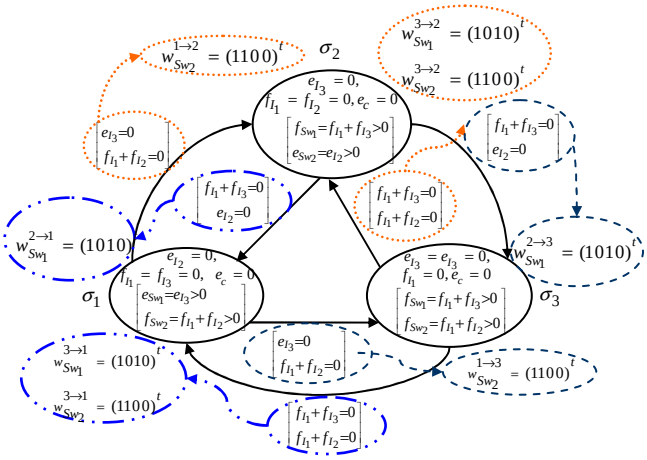

Figure.17. Hybrid automaton

VI. CONCLUSION

In this paper the, controllability of CSLS systems modelled by bond graph was proposed in the form of a procedure containing two main results. Both first ones are sufficient conditions and the third is a necessary and sufficient condition. In all cases graphical interpretations were proposed. These methods are exclusively based on simple causal manipulations on the bond graph model. Our next work consists in studying the observability problem of CSLS systems and extending these results to other classes of hybrid systems.

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