Simulating the Effects of a Non-Uniform Gravitational Field on a Space Robot

Douglas R. Isenberg, Y.P. Kakad
University of North Carolina at Charlotte
Department of Electrical and Computer Engineering,
Charlotte, North Carolina, USA
Email: drisenbe@uncc.edu, kakad@uncc.edu

***Abstract*—This paper investigates the influence that a non-uniform gravitational field has on the dynamics of a space robot. This is accomplished by obtaining the differential equations of motion of the space robot using three gravitational field potential approximations: a uniform field approximation, a zeroth-order Taylor series expansion of the field about the center of mass of each body, and a second-order binomial series expansion of the gravitational field. The three models are then simulated in a free-fall from identical initial conditions. The results indicate that a zeroth-order series expansion of the gravitational field about the center of mass of each body provides a sufficiently high degree of accuracy without resulting in a significant computational burden.**

***Index Terms*—space robotics, dynamical modeling, simulation, gravitational field, MacCullagh formula, Lagrangian mechanics**

I. INTRODUCTION

Research into the dynamics of space robots has been predominately focused on robots that are not under the influence of a gravitational field [1], [2], [3]. Such robots, in which only kinetic forces are present in the dynamics, are generally assumed to be in an orbit. However, it is a non-uniform gravitational field that is responsible for the ability to achieve most orbits. Separating the forces of gravity from the dynamics removes the ability to study more complex phenomena such as the orbit degradation caused by manipulator motion and optimal maneuvering [4].

Preliminary work in this area investigated two methods for incorporating a gravitational field into the model for a space robot each of which leads to motion trajectories that are inconsistent with the trajectory obtained when the gravitational field is neglected [5].

The space robot that is considered in this paper consists of a rigid satellite, or base, and a rigid two-link manipulator as depicted in Fig. 1. In addition to being rigid, each body is assumed to have constant mass. The coordinate frame S is fixed at the center of mass of the satellite. Frame 1 is fixed in link 1 and rotates about an axis parallel to the z-axis. Frame 2 is fixed in link 2 and also rotates about an axis parallel to the z-axis.

Three dynamical models for this space robot are developed. One model is obtained under the assumption that the gravitational field is uniform over the volume spanned by the space robot. This assumption is typical of terrestrial robotic manipulator analysis in which the gravitational force is often the major force in the dynamics. Another model is obtained using a zeroth-order Taylor series approximation of a continuous gravitational field about the center of mass of each body. The other model is obtained by approximating the gravitational field with a second-order binomial series expansion. The differential equations of motion for each model are obtained using the Euler-Lagrange formulation of mechanics.

The resulting equations of motion are extremely cumbersome and require the aid of symbolic computation software. This is discussed. Furthermore, simulations of the three models are carried out by integrating the differential equations of motion with a method that circumvents the problem of symbolically inverting a high dimensional mass matrix.

II. DYNAMICAL MODELING

The Euler-Lagrange formulation of mechanics requires a scaler quantity known as the Lagrangian (1), which is defined as the difference between the kinetic energy, K ,

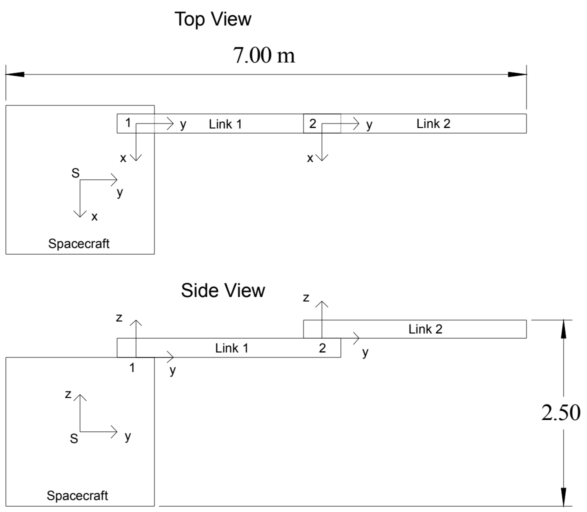

Figure 1. Space robot.

doi:10.4304/jcp.4.12.1255-1262

and the potential energy, U.

$$L = K - U \tag{1}$$

The kinetic energy possessed by a body is a consequence of its motion. The potential energy is typically a consequence of a body's position.

A. Kinetic Energy

The kinetic energy of any multibody system is obtained by taking the superposition of the kinetic energies of each of the bodies that comprise the system. Consider the i^th^ body of a multibody system as depicted in Fig. 2. The position of the k^th^ mass particle, $m_{i,k}$, as measured in the inertial frame, I, is given by:

$$\underline{R_{i,k}} = \underline{R_{b_i}} + {}^{I}T_{b_i} \underline{{}^{b_i}r_{i,k}} . \tag{2}$$

Here, ${}^{I}T_{b_i}$ is a orthogonal transformation which transforms vectors measured with respect to the i^th^ body frame, b_i, into vectors measured with respect to the inertial frame.

Since the body is assumed to be rigid, $\underline{{}^{b_i}r_{i,k}}$ does not change with time. Thus, the velocity of the k^th^ particle in the i^th^ body is obtained as:

$$\underline{\dot{R}_{i,k}} = \underline{\dot{R}_{b_i}} + {}^{I}T_{b_i} (\underline{{}^{b_i}\omega_i} \times \underline{{}^{b_i}r_{i,k}}) , \tag{3}$$

where $\underline{{}^{b_i}\omega_i}$ is the angular velocity vector about which the the body rotates. The square of the magnitude of the velocity is then obtained by computing the inner product of the velocity vector:

$$\begin{aligned} \underline{\dot{R}_{i,k}}^T \underline{\dot{R}_{i,k}} = {} & \underline{\dot{R}_{b_i}}^T \underline{\dot{R}_{b_i}} + 2 \underline{\dot{R}_{b_i}}^T {}^{I}T_{b_i} (\underline{{}^{b_i}\omega_i} \times \underline{{}^{b_i}r_{i,k}}) \\ & + (\underline{{}^{b_i}\omega_i} \times \underline{{}^{b_i}r_{i,k}})^T (\underline{{}^{b_i}\omega_i} \times \underline{{}^{b_i}r_{i,k}}) \end{aligned} . \tag{4}$$

The final term in (4) is also expressed in a matrix quadratic form as:

$$(\underline{{}^{b_i}\omega_i} \times \underline{{}^{b_i}r_{k_i}})^T (\underline{{}^{b_i}\omega_i} \times \underline{{}^{b_i}r_{k_i}}) = \underline{{}^{b_i}\omega_i}^T \hat{J}_{i,k} \underline{{}^{b_i}\omega_i} , \tag{5}$$

where $\hat{J}_{i,k}$ is expressed as:

$$\hat{J}_{i,k} = \begin{bmatrix} {}^{b_i}y_{i,k}^2 + {}^{b_i}z_{i,k}^2 & -{}^{b_i}x_{i,k}\,{}^{b_i}y_{i,k} & -{}^{b_i}x_{i,k}\,{}^{b_i}z_{i,k} \\ -{}^{b_i}x_{i,k}\,{}^{b_i}y_{i,k} & {}^{b_i}x_{i,k}^2 + {}^{b_i}z_{i,k}^2 & -{}^{b_i}y_{i,k}\,{}^{b_i}z_{i,k} \\ -{}^{b_i}x_{i,k}\,{}^{b_i}z_{i,k} & -{}^{b_i}y_{i,k}\,{}^{b_i}z_{i,k} & {}^{b_i}x_{i,k}^2 + {}^{b_i}y_{i,k}^2 \end{bmatrix} . \tag{6}$$

The kinetic energy of the k^th^ particle in the i^th^ body is then obtained from:

$$K_{i,k} = \frac{1}{2} \underline{\dot{R}_{i,k}}^T \underline{\dot{R}_{i,k}} m_{i,k} . \tag{7}$$

Substituting (4) into (7), the kinetic energy of the k^th^ particle is expressed as:

$$\begin{aligned} K_{i,k} = {} & \frac{1}{2} \underline{\dot{R}_{b_i}}^T \underline{\dot{R}_{b_i}} m_{i,k} + \underline{\dot{R}_{b_i}}^T {}^{I}T_{b_i} (\underline{{}^{b_i}\omega_i} \times \underline{{}^{b_i}r_{i,k}} m_{i,k}) \\ & + \frac{1}{2} \underline{{}^{b_i}\omega_i}^T \hat{J}_{i,k} m_{i,k} \underline{{}^{b_i}\omega_i} \end{aligned} . \tag{8}$$

By summing together the kinetic energy of every particle that comprises the i^th^ body, the kinetic energy of the i^th^ body is obtained as:

$$\begin{aligned} K_i = {} & \frac{1}{2} \underline{\dot{R}_{b_i}}^T \underline{\dot{R}_{b_i}} m_i + \underline{\dot{R}_{b_i}}^T {}^{I}T_{b_i} (\underline{{}^{b_i}\omega_i} \times \underline{\Gamma_i}) \\ & + \frac{1}{2} \underline{{}^{b_i}\omega_i}^T J_i \underline{{}^{b_i}\omega_i} \end{aligned} . \tag{9}$$

Here, m_i is defined as the mass of the i^th^ body, $\underline{\Gamma_i}$ is a vector of first moments which is calculated by:

$$\underline{\Gamma_i} = \sum_k \underline{{}^{b_i}r_{i,k}} m_{i,k} , \tag{10}$$

and J_i is the inertia matrix of i^th^ body which is calculated by:

$$J_i = \sum_k \hat{J}_{i,k} m_{i,k} . \tag{11}$$

If the i^th^ body is continuous, then the summations in (10) and (11) simply converge into integrals.

The kinetic energy of the entire system is then obtained by summing together the kinetic energy of each body:

$$K = \sum_i K_i . \tag{12}$$

B. Potential Energy

The derivation of the potential energy of a multibody system follows a similar formulation. Assuming that the Earth is spherical and of uniform density as well as the sole source of a gravitational field centered at the origin of the inertial frame, then the potential energy of the k^th^ particle in the i^th^ body located in the field is obtained as:

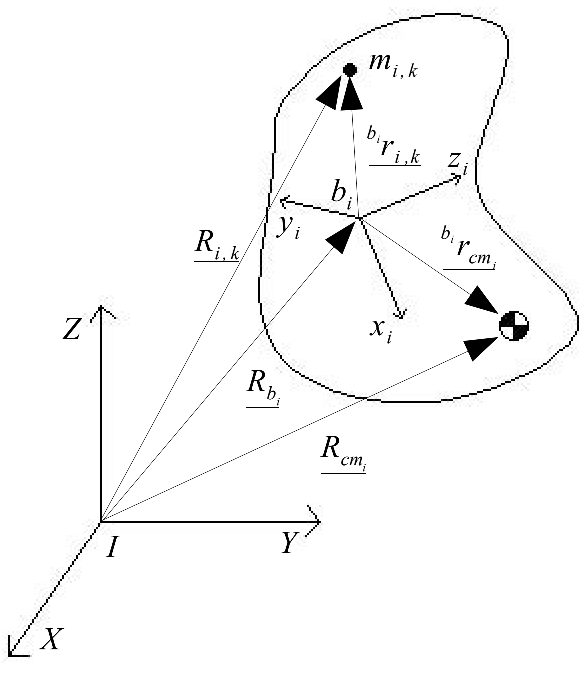

Figure 2. Vector definitions for the i^th^ body.

$$U_{i,k} = \frac{m_{i,k}\,\mu}{\left\|\underline{R_{i,k}}\right\|}\ , \tag{13}$$

where μ is the geocentric gravitational constant [6]. The potential energy of the i[th] body is then:

$$U_i = \sum_k \frac{m_{i,k}\,\mu}{\left\|\underline{R_{i,k}}\right\|}\ . \tag{14}$$

For a continuous body, (14) converges into an integral:

$$U_i = \int_i \frac{\mu\, d\, v}{\left\|\underline{R_i}\right\|}\ . \tag{15}$$

This volume integration poses a significant challenge for bodies that are anything other than the simplest of shapes. It is thus customary to utilize mathematical approximations which simplify the integration.

In a uniform gravitational field, it is well known that the center of mass of a body is also the center of the gravitational force. So, most terrestrial robotic systems are modeled by considering the gravitational force acting on a single particle whose mass is that of the total body [7]. If the variation of the gravitational field over the body is small enough, the gravitational potential of the i[th] body may then be approximated by a zeroth-order Taylor series expansion of (15), about the i[th] center of mass:

$$U_i \approx \frac{\mu\, m_i}{\left\|\underline{R_{cm}}\right\|}\ . \tag{16}$$

Before obtaining a higher order series approximation for the potential energy of the i[th] body, it is convenient to express the denominator of (13) in terms of the center of mass. If a translated version of frame b_i is placed at the center of mass, then from Fig. 2, the following relationship is noticed:

$$\underline{R_{i,k}} = \underline{R_{cm_i}} + {}^{I}T_{b_i}\left(-{}^{b_i}\underline{r_{cm_i}} + {}^{b_i}\underline{r_{i,k}}\right)\ . \tag{17}$$

Letting

$${}^{c}\underline{r_{i,k}} = {}^{I}T_{b_i}\left(-{}^{b_i}\underline{r_{cm_i}} + {}^{b_i}\underline{r_{i,k}}\right)\ , \tag{18}$$

(17) may be expressed using the law of cosines as:

$$\left\|\underline{R_{i,k}}\right\| = \sqrt{\left\|\underline{R_{cm_i}}\right\|^2 + \left\|{}^{c}\underline{r_{i,k}}\right\|^2 - 2\left\|\underline{R_{cm_i}}\right\|\left\|{}^{c}\underline{r_{i,k}}\right\|\cos\theta_{i,k}}\ , \tag{19}$$

where $\theta_{i,k}$ is the angle that is made between $\underline{R_{cm_i}}$ and ${}^{c}\underline{r_{i,k}}$. The denominator of (13) may then be expressed as:

$$\left\|\underline{R_{i,k}}\right\|^{-1} = \left\|\underline{R_{cm_i}}\right\|^{-1}\left(1 + \frac{\left\|{}^{c}\underline{r_{i,k}}\right\|^2}{\left\|\underline{R_{cm_i}}\right\|^2} - 2\frac{\left\|{}^{c}\underline{r_{i,k}}\right\|}{\left\|\underline{R_{cm}}\right\|}\cos\theta_{i,k}\right)^{-\frac{1}{2}}\ . \tag{20}$$

By expanding the bracketed term in (20) with a binomial series, (21) is obtained.

$$\left\|\underline{R_{i,k}}\right\|^{-1} = \left\|\underline{R_{cm_i}}\right\|^{-1}\left(1 - \frac{1}{2}\left(\frac{\left\|{}^{c}\underline{r_{i,k}}\right\|^2}{\left\|\underline{R_{cm_i}}\right\|^2} - 2\frac{\left\|{}^{c}\underline{r_{i,k}}\right\|}{\left\|\underline{R_{cm_i}}\right\|}\cos\theta_{i,k}\right) + \frac{3}{8}\left(\frac{\left\|{}^{c}\underline{r_{i,k}}\right\|^2}{\left\|\underline{R_{cm_i}}\right\|^2} - 2\frac{\left\|{}^{c}\underline{r_{i,k}}\right\|}{\left\|\underline{R_{cm_i}}\right\|}\cos\theta_{i,k}\right)^2 + \cdots\right) \tag{21}$$

If it is assumed that the size of the body is much smaller than the distance from the center of gravitational attraction to the body,

$$\frac{\left\|{}^{c}\underline{r_{i,k}}\right\|}{\left\|\underline{R_{cm_i}}\right\|} \ll 1\ , \tag{22}$$

then (13) is suitably approximated as:

$$U_{i,k} \approx \frac{m_{i,k}\,\mu}{\left\|\underline{R_{cm_i}}\right\|}\left(1 + \frac{\left\|{}^{c}\underline{r_{i,k}}\right\|^2}{\left\|\underline{R_{cm_i}}\right\|^2} + \frac{\left\|{}^{c}\underline{r_{i,k}}\right\|}{\left\|\underline{R_{cm_i}}\right\|}\cos\theta_{i,k} - \frac{3}{2}\frac{\left\|{}^{c}\underline{r_{i,k}}\right\|^2}{\left\|\underline{R_{cm_i}}\right\|^2}\sin^2\theta_{i,k}\right)\ . \tag{23}$$

The potential energy of the i[th] body is then approximated by summing together the potential energy of the k particles that comprise the body.

$$U_i \approx \frac{\mu}{\left\|\underline{R_{cm_i}}\right\|}\sum_k\left(m_{i,k} + \frac{1}{\left\|\underline{R_{cm_i}}\right\|^2}\left\|{}^{c}\underline{r_{i,k}}\right\|\cos\theta_{i,k}\, m_{i,k} + \frac{1}{\left\|\underline{R_{cm_i}}\right\|^2}\left\|{}^{c}\underline{r_{i,k}}\right\|^2 m_{i,k} - \frac{3}{2\left\|\underline{R_{cm_i}}\right\|^2}\left\|{}^{c}\underline{r_{i,k}}\right\|^2\sin^2\theta_{i,k}\, m_{i,k}\right) \tag{24}$$

After distributing the summation in (25) and simplifying, the second term vanishes and the gravitational potential of the i[th] body is conveniently expressed as:

$$U_i \approx \frac{\mu}{\left\|\underline{R_{cm_i}}\right\|}\left(m_i + \frac{trace\left({}^{c}J_i\right)}{2\left\|\underline{R_{cm_i}}\right\|^2} - \frac{3}{2\left\|\underline{R_{cm_i}}\right\|^4}\underline{R_{cm_i}^T}\,{}^{I}T_{b_i}\,{}^{c}J_i\,{}^{I}T_{b_i}^T\,\underline{R_{cm_i}}\right)\ . \tag{25}$$

This expression is referred to as the MacCullagh formula [8], [9]. It should be noticed that the first term in (25) is the same as (16). The two additional terms may then be thought of as correctional terms. Also, it should be noticed that the inertia matrices in (25) are measured with respect to the translated version of frame b_i which is located at the center of mass of the i[th] body. In general, these inertia matrices will be different from the inertia matrices used in the calculation of the kinetic energy unless the body frames are initially located at the center of mass such that the magnitude of the translation is zero.

The potential energy of the entire space robot is then obtained from the superposition of the potential energy of each body:

$$U = \sum_i U_i \,. \tag{26}$$

Equations (16) and (25) provide different potential energy approximations for bodies in a non-uniform gravitational field. If a field is uniform over the volume of the robot, then an additional step must be taken. A point must be selected from which the strength of the field over the volume can be determined. If that point is chosen to be the position of the satellite, then the potential energy of the space robot is obtained as:

$$U \approx \sum_i \frac{\mu\, m_i \left\| \underline{R_{cm_i}} \right\|}{\left\| \underline{R_{cm_1}} \right\|^2} \,. \tag{27}$$

C. System Parameters

The location of the satellite body frame for the space robot depicted in Fig. 1 is given as:

$$\underline{R_{b_1}} = [x \quad y \quad z]^T \,. \tag{28}$$

The transformation that transforms vectors measured with respect to the satellite's body frame into vectors measured with respect to the inertial frame is given by the Euler-angle rotation sequence parameterized in (29).

$${}^I T_{b_1} = \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ s_\phi s_\theta c_\psi + c_\phi s_\psi & -s_\phi s_\theta s_\psi + c_\phi c_\psi & -s_\phi c_\theta \\ -c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi + s_\phi c_\psi & c_\phi c_\theta \end{bmatrix} \tag{29}$$

Here, ϕ denotes the angle of vector rotation about the x-axis, θ denotes the angle of vector rotation about the new y-axis, and ψ denotes the the angle of vector rotation about the new z-axis.

The transformation that transforms vectors measured with respect to the first link's body frame into vectors measured with respect to the inertial frame is given as:

$${}^I T_{b_2} = {}^I T_{b_1} \begin{bmatrix} c_{\gamma_1} & -s_{\gamma_1} & 0 \\ s_{\gamma_1} & c_{\gamma_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{30}$$

where γ_1 is the angular position of the first link as measured with respect to b_1.

The transformation that transforms vectors measured with respect to the second link's body frame into vectors measured with respect to the inertial frame is given by:

$${}^I T_{b_3} = {}^I T_{b_2} \begin{bmatrix} c_{\gamma_2} & -s_{\gamma_2} & 0 \\ s_{\gamma_2} & c_{\gamma_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{31}$$

where γ_2 is the angular position of the second link as measured with respect to b_2.

The angular velocity of the satellite, as measured with respect to frame b_1, is given as:

$$\underline{{}^{b_1}\omega_1} = \begin{bmatrix} s_\psi \dot\theta + c_\psi c_\theta \dot\phi \\ c_\psi \dot\theta - s_\psi c_\theta \dot\phi \\ s_\theta \dot\phi + \dot\psi \end{bmatrix} . \tag{32}$$

The angular velocity of the first link, as measured with respect to frame b_2, is given by:

$$\underline{{}^{b_2}\omega_2} = \begin{bmatrix} \dot\theta\, s_{(\gamma_1+\psi)} + \frac{1}{2}\dot\phi\, c_{(\gamma_1+\psi+\theta)} + \frac{1}{2}\dot\phi\, c_{(\gamma_1+\psi-\theta)} \\ \dot\theta\, c_{(\gamma_1+\psi)} - \frac{1}{2}\dot\phi\, s_{(\gamma_1+\psi+\theta)} - \frac{1}{2}\dot\phi\, s_{(\gamma_1+\psi-\theta)} \\ \dot\phi\, s_\theta + \dot\gamma_1 + \dot\psi \end{bmatrix}, \tag{33}$$

and the angular velocity of the second link, as measured with respect to frame b_3, is given by:

$$\underline{{}^{b_3}\omega_3} = \begin{bmatrix} \left(\frac{1}{2}\dot\phi\, c_{(\gamma_2+\gamma_1+\psi-\theta)} + \frac{1}{2}\dot\phi\, c_{(\gamma_2+\gamma_1+\psi+\theta)} \right. \\ \left. + \dot\theta\, s_{(\gamma_2+\gamma_1+\psi)}\right) \\ \left(-\frac{1}{2}\dot\phi\, s_{(\gamma_2+\gamma_1+\psi+\theta)} - \frac{1}{2}\dot\phi\, s_{(\gamma_2+\gamma_1+\psi-\theta)} \right. \\ \left. + \dot\theta\, c_{(\gamma_2+\gamma_1+\psi)}\right) \\ \dot\gamma_1 + \dot\psi + s_\theta \dot\phi + \dot\gamma_2 \end{bmatrix} . \tag{34}$$

The inertia matrix of the satellite is:

$$J_1 = {}^c J_1 = 16/3\; I_{3\times 3} \,. \tag{35}$$

The inertia matrix of the first link and the second link, as measured in frames b_1 and b_2, is:

$$J_2 = J_3 = \begin{bmatrix} 7/16 & 0 & 0 \\ 0 & 5/1024 & -15/512 \\ 0 & -15/512 & 445/1024 \end{bmatrix} . \tag{36}$$

The inertia matrix of the first link and the second link, as measured with respect to the center of mass of the first link and second link, is given by:

$${}^c J_2 = {}^c J_3 = \begin{bmatrix} 145/1024 & 0 & 0 \\ 0 & 1/512 & 0 \\ 0 & 0 & 145/1024 \end{bmatrix} \tag{37}$$

The first mass moments are given by:

$$\begin{gathered} \underline{\Gamma_1} = [0 \quad 0 \quad 0]^T \\ \underline{\Gamma_2} = \underline{\Gamma_3} = [0 \quad 15/64 \quad 3/128]^T \end{gathered} , \tag{38}$$

and the mass of each body is given as:

$$\begin{gathered} m_1 = 8 \\ m_2 = m_3 = 3/16 \end{gathered} . \tag{39}$$

Furthermore, the following vectors are also required:

$$\begin{aligned} {}^{b_1}\underline{r}_{b_2} &= [-.75 \quad .75 \quad 1]^T \\ {}^{b_2}\underline{r}_{b_3} &= [0 \quad 2.5 \quad 0.25]^T \\ {}^{b_2}\underline{r}_{cm_2} &= [0 \quad 1.25 \quad 0.125]^T \\ {}^{b_3}\underline{r}_{cm_3} &= [0 \quad 1.25 \quad 0.125]^T \end{aligned} \quad (40)$$

III. Differential Equations of Motion

The differential equations of motion are obtained through the use of the Euler-Lagrange formulation of mechanics. After forming the Lagrangian (1), the equations of motion are obtained by applying Euler's equation:

$$\underline{F} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{q}}}\right) - \frac{\partial L}{\partial \underline{q}} . \quad (41)$$

In (41), $\underline{q}$ is a vector of the system's generalized coordinates and $\underline{F}$ is a vector of the input forces and torques in the direction of the generalized coordinates. It is guaranteed that the system's coordinates will be generalized if the number of coordinates specifying the position of the system is equal to the number of degrees of freedom of the system. This is easily accomplished when $\underline{R}_{b_1}$ is specified in Cartesian coordinates and the transformations, ${}^{i}T_{i-1}$, are specified in Euler-angles.

The challenge in obtaining the differential equations of motion from (41) stems from the extremely large size of the resulting equations. While the differentiations in (41) may be performed manually for simple systems, it is not practical for multibody systems in three dimensions. Symbolic manipulation programs, on the other hand, are well adapted to such as task.

For nearly all systems, the differential equations of motion that are obtained from (41) will take on the following form:

$$\underline{F} = H(\underline{q})\ddot{\underline{q}} + \underline{D}(\underline{q},\dot{\underline{q}}) + \underline{G}(\underline{q}) . \quad (42)$$

Here, $H(\underline{q})$ is the system's mass matrix, $\underline{D}(\underline{q},\dot{\underline{q}})$ is a vector that contains the Coriolis and centrifugal forces and torques, and $\underline{G}(\underline{q})$ is a vector that contains the gravitational forces and torques. The symbolic expressions for these terms are obtained after reexamining the right hand side of (41). Expanding the Lagrangian, (41) becomes:

$$\underline{F} = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\underline{q}}}\right) - \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{\underline{q}}}\right) - \frac{\partial K}{\partial \underline{q}} + \frac{\partial U}{\partial \underline{q}} . \quad (43)$$

Since the potential energy is a function of position only, the second term vanishes. Immediately, it is recognized that:

$$\underline{G}(\underline{q}) = \frac{\partial U}{\partial \underline{q}} . \quad (44)$$

Thus, the matrix containing the gravitational forces is obtained through a simple symbolic partial differentiation of the potential energy.

It is known that the time-derivative of a function of time-varying variables may be obtained through the use of the Jacobian matrix of the function. Since the Jacobian operator is common to many symbolic math programs, the major hurdle in symbolically computing (41) is rather easily overcome. The first term on the right hand side of (43) is expressed, in terms of the Jacobian as as:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\underline{q}}}\right) = Jacobian\left(\frac{\partial K}{\partial \dot{\underline{q}}}, [\underline{q}^T \, \dot{\underline{q}}^T]\right)\begin{bmatrix}\dot{\underline{q}} \\ \ddot{\underline{q}}\end{bmatrix} . \quad (45)$$

The first term in this Jacobian operator notation indicates the particular function that is to be operated on. The second term indicates the variables through which the Jacobian is formed . By partitioning the Jacobian matrix into the portion that is multiplied by $\ddot{\underline{q}}$, the mass matrix is symbolically obtained as:

$$H(\underline{q}) = Jacobian\left(\frac{\partial K}{\partial \dot{\underline{q}}}, [\dot{\underline{q}}^T]\right) . \quad (46)$$

The vector of Coriolis and centrifugal forces must then be comprised of the remaining terms in (43):

$$\underline{D}(\underline{q},\dot{\underline{q}}) = Jacobian\left(\frac{\partial K}{\partial \dot{\underline{q}}}, [\underline{q}^T]\right)\dot{\underline{q}} - \frac{\partial K}{\partial \underline{q}} . \quad (47)$$

By solving (42) for the accelerations, one obtains:

$$\ddot{\underline{q}} = H(\underline{q})^{-1}(\underline{F} - \underline{D}(\underline{q},\dot{\underline{q}}) - \underline{G}(\underline{q})) . \quad (48)$$

The symbolic expression for the accelerations of the generalized coordinates is possible with the substitution of (44), (46), and (47) into equation (48). Unfortunately, this leads to an expression that is unmanageable due to its extreme size. Eventually, (48) must be converted into a system of first-order differential equations if it is to be integrated using a iterative scheme such as Runge-Kutta. The conversion is accomplished by assigning state variables. By letting the new states be denoted by α_i , the differential equations of motion are expressed as a first order system of equations through the following state variable assignments:

$$\begin{aligned} \dot{\alpha}_1 &= \alpha_2 \\ \dot{\alpha}_2 &= \ddot{q}_1 \\ \dot{\alpha}_3 &= \alpha_4 \\ \dot{\alpha}_4 &= \ddot{q}_2 \\ &\vdots \end{aligned} \quad (49)$$

The equations of motion may then be solved by first expressing (44), (46), and (47) in terms of the state variables. Then, for each iteration of the integration, the numerical values of (44), (46), and (47) are obtained by substitution of the numerical states into the expressions. This may be accomplished by creating three functions or subroutines that are separate from the integration but called upon during each iteration. The numerical solutions of (44), (46), and (47) are then substituted into (48) along with $\underline{F}$. The accelerations in terms of state variables for that single iteration are then obtained. By repeating this process for all iterations of the integration, the position and velocity evolutions are obtained.

This approach is particularly attractive when the mass matrix has a very high dimensionality, making the symbolic computation of its inverse impractical. In general, the mass matrix is invertible except when the system is located near a singularity. Care should always be taken in this portion of the computation to ensure sufficient accuracy.

IV. Simulations and Analysis

A forth-order fixed step size Runge-Kutta scheme is used to integrate the differential equations of motion of the three models. By using a step size of $0.0001\,s$, the truncation error is proportional to 1×10^{-20} [10]. This has been indicated by practice to be sufficient for this problem.

In each simulation, the robot is allowed to drop from an altitude of $39810\times10^{3}\ m$ for $20\,s$. Other than the initial z-coordinate of the satellite which serves as the initial altitude, all other initial conditions are zeroed.

The time evolution of the space robot's generalized coordinates, when modeled in a uniform gravitational field, is presented in Fig. 3. It is seen that fairly large motion is induced by the gravitational field. Particularly interesting is the hint of chaotic motion of the manipulator which should be expected from a multi-pendulum type system in a uniform gravitational field.

The time evolution of the space robot's generalized coordinates when modeled with the two non-uniform gravitational field approximations is presented in Fig. 4. Interestingly, there appears to be no-significant difference between the trajectories predicted by the two different models. Additionally, both trajectories are extremely different from the trajectories obtained from the model that assumes a uniform gravitational field. The large angle rotations and significant translations that are depicted in Fig. 3 are not present in the trajectories of Fig. 4.

The MacCullagh formula, (25), adds two additional higher order terms to the zeroth-order Taylor series approximation of (16) for each body. As mentioned, these may be viewed as correctional terms. Any discrepancies between the time evolutions of the system coordinates of the two models must be attributed to those terms. Naturally, the quantitative effects of the terms are of interest. The forces and torques due to the potential energy are found by evaluating the equations obtained from (44) with the data obtained from the simulations. The forces and torques that result from the correctional terms are obtained by subtracting away the forces and

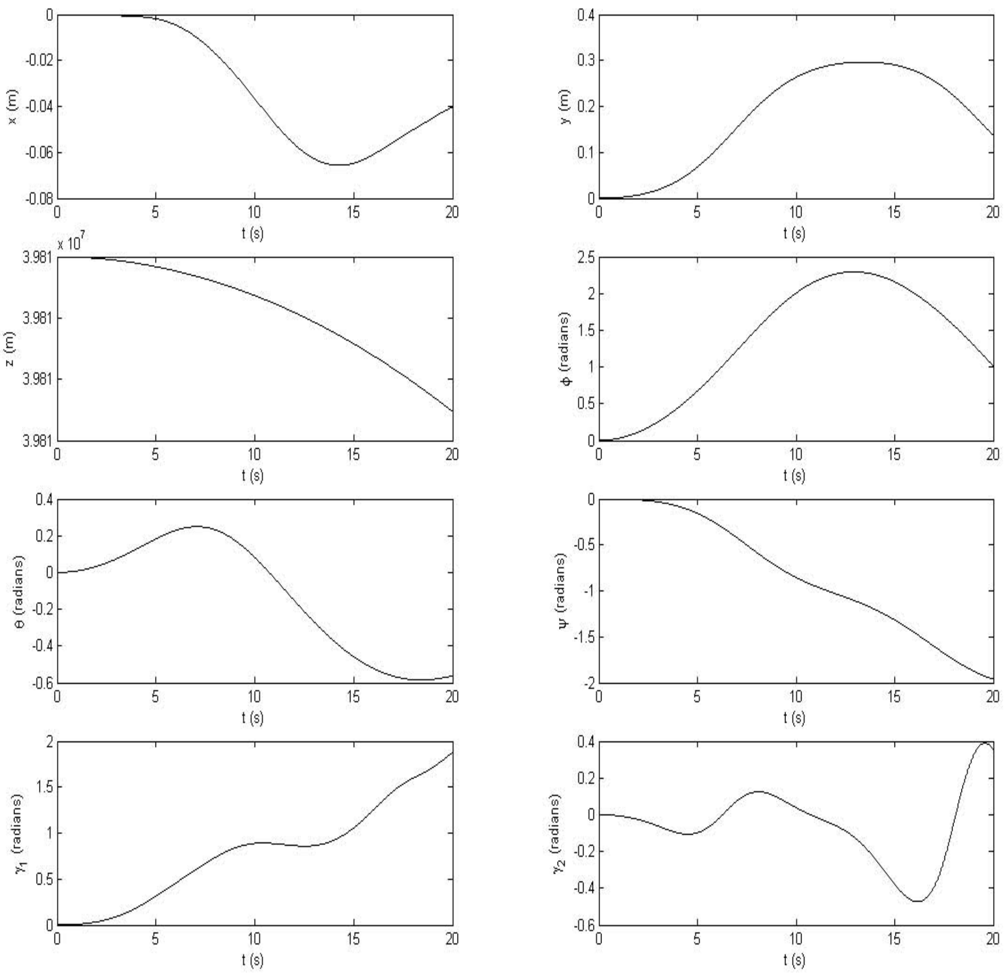

Figure 3. Time evolution of system coordinates when using a uniform gravitational field approximation.

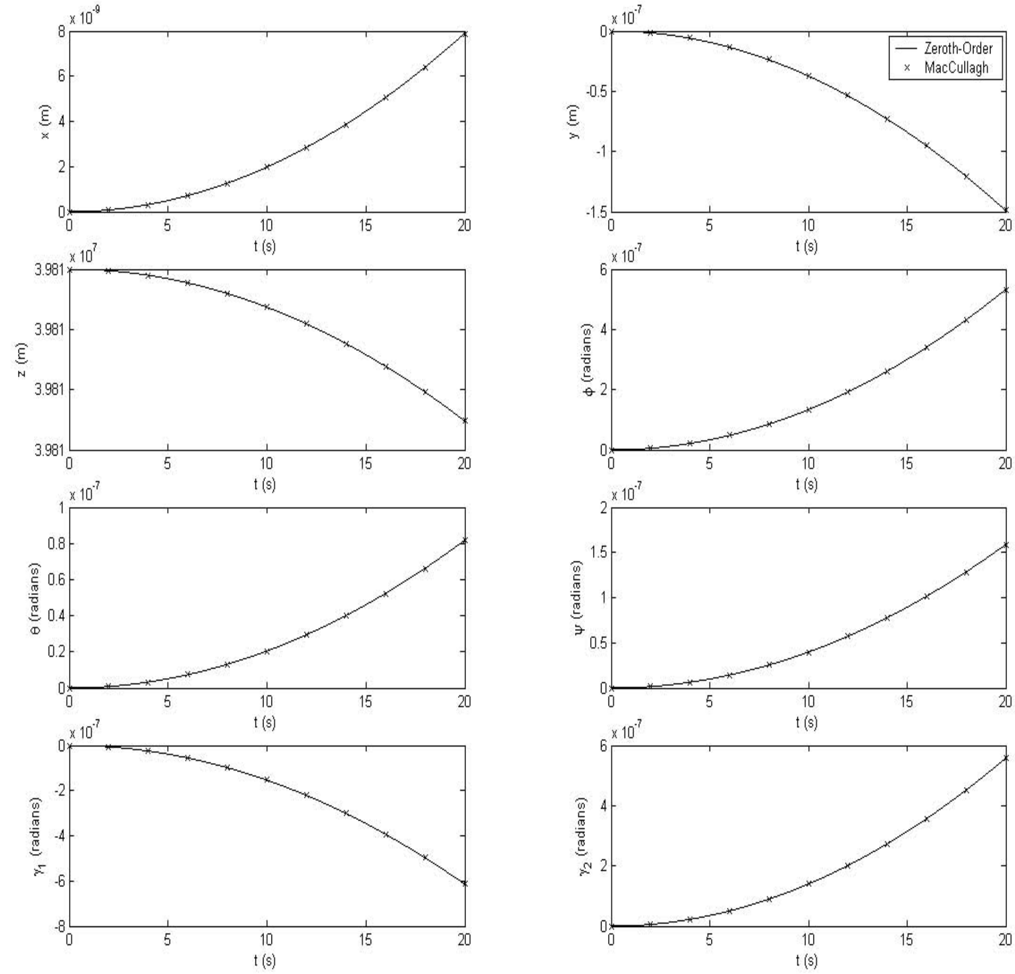

Figure 4. Time evolution of system coordinates when using non-uniform gravitational field approximations

torques that result from the zeroth-order term. The solid lines in Fig. 5 display the evolution of the forces and torques that are due to the zeroth-order term. They are measured on the left hand axes. The dashed lines in Fig. 5 display the evolution of the forces and torques that result from the correctional terms of the MacCullagh formula. These are measured on the right hand axes. The correctional forces and torques are denoted with the subscript, c .

Despite the poor resolution, one important piece of information is gained from Fig. 5. The forces and torques caused by the correctional terms in the MacCullagh formula contribute very little to the dynamics. The zeroth-order term, in contrast, influences the dynamics significantly. Nearly a dozen orders of magnitude constitute the difference.

V. Conclusion

This paper explores the effects of a non-uniform gravitational field on the dynamics of a space robot. First, the kinetic energy of a two link space robot is developed. Three gravitational potential approximations are then made. Methods for symbolically obtaining and numerically integrating the differential equations of motion is discussed. Then the differential equations of motion are simulated. It is shown that a uniform field approximation predicts a different trajectory than the more mathematically accurate zeroth-order Taylor series and MacCullagh formula approximations of the continuous gravitational potential.

The correctional terms of the MacCullagh formula contribute very little to the dynamics at the simulated altitude. They do, however, contribute significantly to the complexity of the differential equations of motion. This complexity increases the time required to perform simulations and hinders the usefulness of the model for analytical analysis and control system design.

At sufficiently high altitudes, or consequently for a sufficiently small body, the zeroth-order Taylor series expansion of a non-uniform gravitational potential about the center of mass of each body provides a simple yet very accurate method for including the effects of gravity into the dynamical equations of motion for a space robot. At lower altitudes, or for larger bodies, the additional accuracy of the MacCullagh formula should prove more useful.

Acknowledgement

This work was carried out under funding from the North Carolina Space Grant Consortium.

References

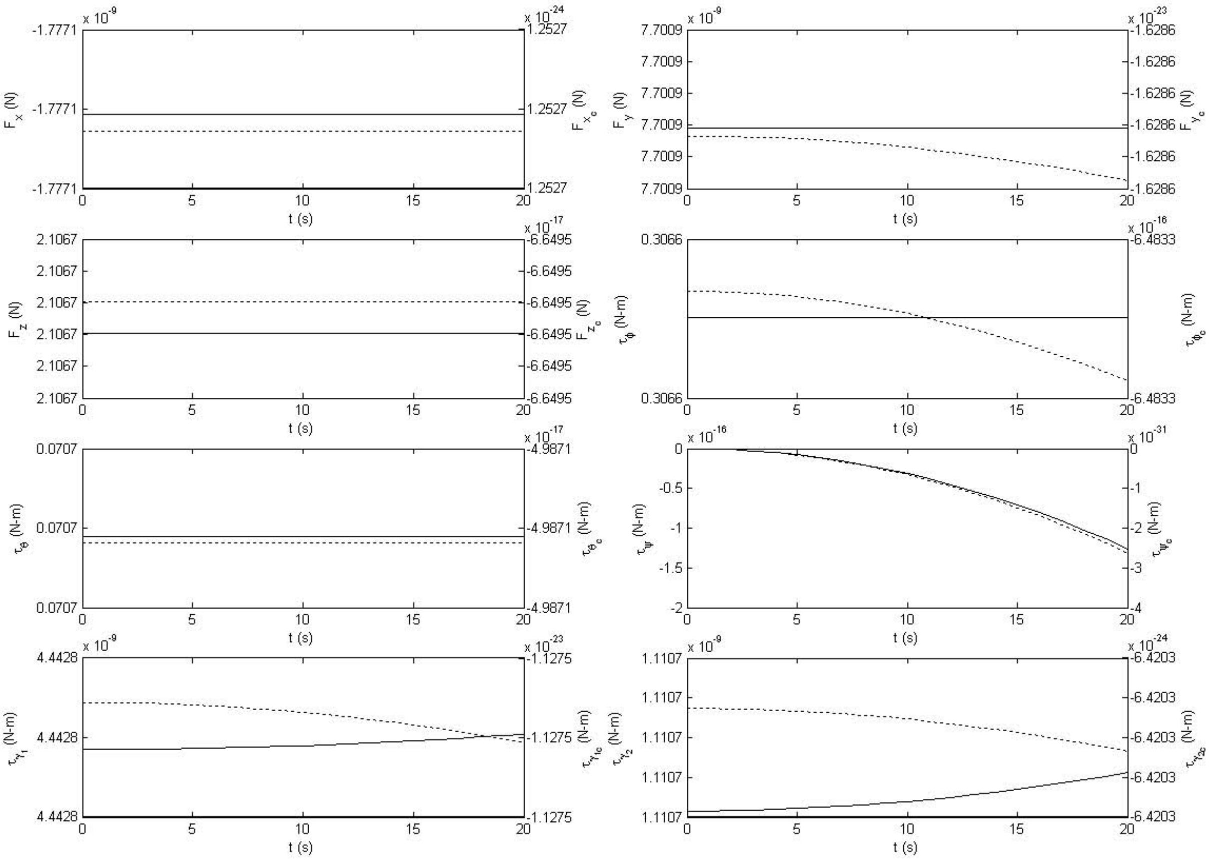

Figure 5. Forces and torques caused by zeroth-order term (left) and correction terms (right).

[1] S. Moosavian, E. Papadopoulos, "Explicit dynamics of space free-flyers with multiple manipulators via SPACEMAPLE," *Advanced Robotics*, vol. 18, no. 2, pp. 223-244, 2004

[2] Z. Vafa, S. Dubowsky, "On the dynamics of manipulators in space using the virtual manipulator approach," *Proceedings: IEEE International Conference on Robotics and Automation*, March 1987

[3] L. Meirovitch, Y. Chen, "Trajectory and control optimization for flexible space robots," *Journal of Guidance, Control, and Dynamics*, vol. 18, no. 3, pp. 493-502, 1995

[4] H. Hablani "Attitude dynamics of a rotating chain of rigid bodies in a gravitational field", *Journal of Guidance*, vol. 8, no. 4, pp. 471-477, 1985

[5] D.R. Isenberg "Dynamics and simulation of a space robot in a gravitational field," *Proceedings: IEEE Southeast Conference*, April 2008

[6] V.R. Bond, M.C. Allman, *Modern Astrodynamics: Fundamentals and Perturbation Methods*, Princeton University Press, Princeton, NJ, 1996

[7] R.P. Paul, *Robot Manipulators: Mathematics, Programming, and Control*, MIT Press, Cambridge, Ma, 1981

[8] P.C. Hughes, *Spacecraft Attitude Dynamics*, Dover, Mineola, NY, 2004

[9] D.A. Vallado, *Fundamentals of Astrodynamics and Applications*, 2nd edition, Microcosm Press, El Segundo California, 2004

[10] Kress, *Numerical Analysis*, Springer, New York, NY, 1998

Douglas R. Isenberg received his B.S. in electrical engineering in 2004 and M.S. in electrical engineering in 2006 from the University North Carolina at Charlotte, USA. He is currently pursuing a Ph.D. in electrical engineering at the University of North Carolina at Charlotte. His research interest includes the dynamical modeling and control of multibody systems. He is a member of IEEE and AIAA.

Y.P. Kakad received his Ph.D. degree in 1975 from the University of Florida. He joined the department of Electrical and Computer Engineering at the University of North Carolina at Charlotte in 1976 where he currently holds the rank of Professor and Interim Chair. His research interests include multivariable controls, robotics, application of control theory to aircrafts and spacecrafts, and digital signal processing.