Computation of the Normalized Detection Threshold for the FFT Filter Bank-Based Summation CFAR Detector

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***Abstract*—The FFT filter bank-based summation CFAR detector is widely used for the detection of narrowband signals embedded in wideband noise. The simulation and implementation of this detector involves some problems concerning the reliable computation of the normalized detection threshold for a given probability of false alarm. This paper presents a comprehensive theoretical treatment of major aspects of the numerical computation of the normalized detection threshold for an AWGN channel model. Equations are derived for the probability of false alarm, P_{fa}, for both non-overlapped and overlapped input data and then used to compute theoretical upper and lower bounds for the detection threshold T. A very useful transformation is introduced that guarantees the global quadratic convergence of the Newton-Ralphson algorithm in the computation of T for overlapped data with an overlap ratio not exceeding 50%. It is shown that if the product of the number of FFT bins assigned to a channel for signal power estimation and the number of input data blocks is relatively small, e.g., less than 60, the theoretical normalized detection threshold can be accurately computed without numerical problems. To handle other cases, good approximations are derived.**

***Index Terms*—Digital FFT filter bank, Detection and estimation, Constant false alarm rate (CFAR) detection.**

I. INTRODUCTION

The FFT filter bank-based summation CFAR detector is an efficient technique for the detection of narrowband signals embedded in wideband noise and has important applications in civilian spectrum monitoring, electronic warfare, radio astronomy and instrumentation [1]-[7]. This detector operates by adding estimates of spectral power of FFT bins in channels, each of which corresponds to a group of one or more contiguous FFT bins, and comparing the sum of the channel power estimates over multiple input data blocks against a detection threshold, T. In its general form, this detector has been the subject of many recent studies [7]-[14]. In particular, closed-form algebraic formulas for computing the probability of false alarm, P_{fa}, as a function of T have been derived [5], [7], [13]. These results can be used to compute the normalized detection threshold (to be defined later) using numerical procedures such as the Newton-Ralphson algorithm. As practical implementations of these numerical procedures require good lower and upper bounds of T for reliable initialization, several theoretical lower and upper bounds for T for a given P_{fa} have also been obtained [8]. However, a self-contained comprehensive treatment of the FFT filter bank-based summation CFAR detector has not appeared in the literature and details of relevant mathematical derivations have not been published.

This paper presents a comprehensive theoretical treatment of major aspects of the numerical computation of the normalized detection threshold for an AWGN channel model. Formulas for computing P_{fa} as a function of T are derived from first principles. Several theoretical lower and upper bounds for T are derived and a very useful transformation is introduced that guarantees the global quadratic convergence of the Newton-Ralphson algorithm in the computation of the normalized detection threshold for overlapped data. Finally, accurate approximations for T are presented. The results presented in this paper are very useful for the simulation and implementation of the FFT filter bank-based summation CFAR detector.

This paper is organized as follows. Section II introduces the FFT filter bank-based summation CFAR detector. Section III derives the formulas for computing P_{fa} for a given T for overlapped and non-overlapped signal data. Section IV proves several lower and upper bounds of T for overlapped and non-overlapped input data. Section V introduces a very useful transformation that guarantees the global quadratic convergence of the Newton-Ralphson algorithm while Section VI derives two approximations for T. Finally Section VII presents concluding remarks. Throughout this paper, typical numerical examples are given to illustrate the results.

II. THE FFT FILTER BANK-BASED SUMMATION CFAR DETECTOR

Consider a uniformly sampled band-limited signal embedded in additive white Gaussian noise. Assume that M

channels are uniformly distributed across the frequency range contained within the Nyquist bandwidth and that K FFT bins are assigned to each channel with the N FFT bins ($N \leq K$) centered within each channel used to estimate the power contained within the channel. Consequently, an FFT of length MK is used to compute the power levels for the M channels. For notational convenience and without loss of generality, assume $K-N$ is an even integer. Let $\mathbf{w} = [w_0, \cdots, w_{MK-1}]^t$ be a symmetric window of length MK, where the superscript t denotes matrix transposition. Let $L \geq 1$ be a positive integer and consider L consecutive overlapping sample vectors, $\mathbf{R}_l$, constructed as follows:

$$\mathbf{R}_l = [r_{l(1-\gamma)MK+MK-1},\ \cdots,\ r_{l(1-\gamma)MK}]^t, \qquad (1)$$

where $0 \leq l \leq L-1$, $0 \leq \gamma \leq 1/2$, r_n is the n-th sample of the input data stream and γMK is an integer. In practice, the overlap ratio, γ, is often selected to be either 0 or $1/2$. These two cases correspond to zero and 50% overlap, respectively. For each l, the two input vectors, $\mathbf{R}_l$ and $\mathbf{R}_{l+1}$, have γMK samples in common. The vectors, $\mathbf{R}_l$, are windowed by $\mathbf{w}$, resulting in the windowed sample vectors:

$$\mathbf{X}_l = [w_0 r_{l(1-\gamma)MK+MK-1},\ \cdots,\ w_{MK-1} r_{l(1-\gamma)MK}]^t.$$

The vectors, $\mathbf{X}_l$, are then transformed by the inverse discrete Fourier transform matrix $\mathbf{F}$ of dimensions $MK \times MK$ to yield the FFT filter bank output sample vectors:

$$\mathbf{Y}_l = \mathbf{F}\mathbf{X}_l = [s_{l,0},\ s_{l,1},\ \cdots,\ s_{l,MK-1}]^t,$$

where

$$\mathbf{F} = \qquad (2)$$
$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & e^{\frac{2\pi jl}{MK}} & \cdots & e^{\frac{2\pi j(MK-1)l}{MK}} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & e^{\frac{2\pi j(MK-1)}{MK}} & \cdots & e^{\frac{2\pi j(MK-1)(MK-1)}{MK}} \end{bmatrix}.$$

From each vector, $\mathbf{Y}_l$, a vector $\mathbf{z}_l = [z_{l,0}, \cdots, z_{l,M-1}]^t$ of length M is formed by summing the power from the N FFT bins centered within each channel:

$$z_{l,k} = \sum_{m=0}^{N-1} |s_{l,I_k+m}|^2,$$
$$0 \leq l \leq L-1,\ 0 \leq k \leq M-1, \qquad (3)$$

i.e., the power estimates from the N FFT bins with indices I_k, I_k+1, $\cdots$, $I_k+(N-1)$ are summed to form the power estimate for the k-th channel for the data block $\mathbf{R}_l$ where $I_k = kK + \frac{K-N}{2}$. The detection criterion is defined as follows: if for a given detection threshold, T, $\sum_{l=0}^{L-1} z_{l,k} \geq T$, a signal is declared to exist in the k-th channel. The choice of T is dependent on the desired probability of false alarm, P_{fa}, the noise spectral density, and the implementation details of the detector.

For brevity, the FFT filter bank-based L-block summation CFAR detector shall be referred to as the L-block summation CFAR detector or the summation CFAR detector or simply the summation detector if there is no risk of confusion. A block diagram is given in Figure 1 for the special case where N and K are equal.

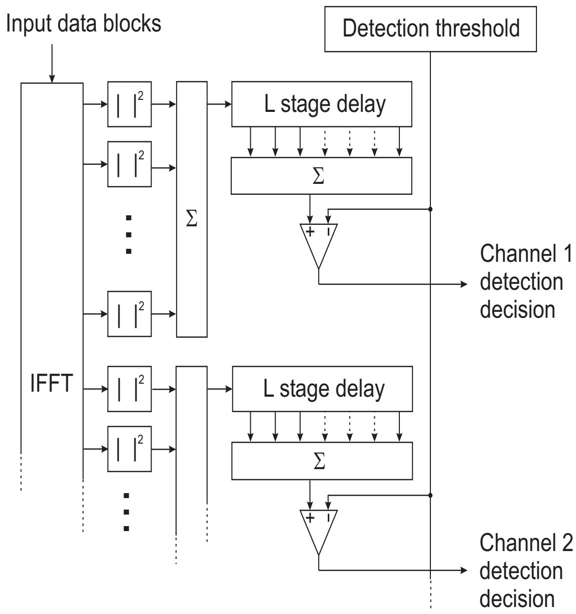

Fig. 1. Block diagram for the L-block summation CFAR detector.

III. The Probability of False Alarm for the Summation CFAR Detector

Assume the input data stream r_n is a zero-mean complex-valued white Gaussian noise sequence with $E(r_p r_q^*) = \sigma^2 \delta_{pq}$, where $\sigma^2 > 0$ is the noise variance (noise floor), $\delta_{pq} = 1$ if $p = q$ and $\delta_{pq} = 0$ if $p \neq q$. For a given threshold, T, the probability of false alarm, P_{fa}, for the k-th channel of the L-block summation CFAR detector is then given by:

$$P_{fa} = \Pr\left\{\sum_{l=0}^{L-1} z_{l,k} \geq T\right\}. \qquad (4)$$

The following three theorems provide the mathematical foundation for computing the threshold T for a given P_{fa}.

Theorem 1. Assume $L \geq 2$ and $0 < \gamma \leq 1/2$. For a given $T > 0$, P_{fa} is given by:

$$P_{fa} = \sum_{l=1}^{LN} \frac{\lambda_l^{LN-1}}{\prod_{1 \leq m \leq LN, m \neq l} (\lambda_l - \lambda_m)} e^{-\frac{T}{\sigma^2 \lambda_l}}, \qquad (5)$$

where λ_l, $1 \leq l \leq LN$, are the LN distinct positive eigenvalues of the $L \times L$ block matrix $\mathbf{H}$:

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^H & \mathbf{A} & \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^H & \mathbf{A} & \mathbf{B} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}^H & \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{B}^H & \mathbf{A} \end{bmatrix}. \qquad (6)$$

In (6), $\mathbf{H}$ is of dimensions $LN \times LN$, $\mathbf{0}$ is the $N \times N$ zero matrix and $\mathbf{A}$ and $\mathbf{B}$ are $N \times N$ matrices defined respectively by:

$$\mathbf{A} = \begin{bmatrix} \tau_{11} & \tau_{12} & \cdots & \tau_{1q} & \cdots & \tau_{1N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \tau_{p1} & \tau_{p2} & \cdots & \tau_{pq} & \cdots & \tau_{pN} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \tau_{N1} & \tau_{N2} & \cdots & \tau_{Nq} & \cdots & \tau_{NN} \end{bmatrix}, \tag{7}$$

$$\tau_{pq} = \sum_{l=0}^{MK-1} w_l^2 \exp \frac{2\pi j l(p-q)}{MK}, \tag{8}$$

$$\mathbf{B} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1q} & \cdots & \gamma_{1N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{p1} & \gamma_{p2} & \cdots & \gamma_{pq} & \cdots & \gamma_{pN} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{N1} & \gamma_{N2} & \cdots & \gamma_{Nq} & \cdots & \gamma_{NN} \end{bmatrix}, \tag{9}$$

$$\gamma_{pq} = e^{-2\pi j(1-\gamma)(I_k+q-1)} \times \sum_{l=0}^{\gamma MK-1} w_l w_{l+(1-\gamma)MK} \exp \frac{2\pi j l(p-q)}{MK}. \tag{10}$$

Theorem 2. Assume $L \geq 1$, $N > 1$ and $\gamma = 0$. For a given $T > 0$, P_{fa} is given by:

$$P_{fa} = \sum_{m=1}^{N} \sum_{p=1}^{L} A_{mp} \sum_{s=0}^{p-1} \frac{\left(\frac{T}{\sigma^2 \mu_m}\right)^s}{s!} e^{-\frac{T}{\sigma^2 \mu_m}}, \tag{11}$$

where the coefficients A_{mp}, $1 \leq m \leq N$, $1 \leq p \leq L$, are defined by:

$$\begin{cases} A_{mL} = \left[\dfrac{\mu_m^{N-1}}{\prod\limits_{1 \leq l \leq N, l \neq m} (\mu_m - \mu_l)} \right]^L, \\ A_{mp} = A_{mL} \times \sum\limits_{k_1+\cdots+k_{m-1}+k_{m+1}+\cdots+k_N = L-p} \Gamma_m(k_1, \cdots, k_{m-1}, k_{m+1}, \cdots, k_N), \\ \Gamma_m(k_1, \cdots, k_{m-1}, k_{m+1}, \cdots, k_N) = \prod\limits_{1 \leq l \leq N, l \neq m} \dfrac{(L+k_l-1)!}{(k_l)!(L-1)!} \left[\dfrac{\mu_l}{\mu_l - \mu_m} \right]^{k_l}. \end{cases} \tag{12}$$

Note that k_q, $1 \leq q \leq N$, are non-negative integers and μ_q, $1 \leq q \leq N$, are the N distinct positive eigenvalues of the Hermitian matrix $\mathbf{A}$ defined by (7).

Theorem 3. Let $L \geq 1$, $N = 1$ and $\gamma = 0$. For a given $T > 0$, P_{fa} is given by:

$$P_{fa} = \sum_{s=0}^{L-1} \frac{\left(\frac{T}{\lambda}\right)^s}{s!} e^{-\frac{T}{\lambda}}, \tag{13}$$

where

$$\lambda = \sigma^2 \sum_{l=0}^{MK-1} w_l^2. \tag{14}$$

The proofs of these theorems are relatively lengthy and will be given in the appendix.

In practical systems, the noise floor, σ^2, needs to be estimated from the output samples of the FFT filter bank. For a given P_{fa}, the detection threshold T is obtained by multiplying the estimated noise floor σ^2 by T/σ^2, which is computed using one of the equations (5), (11) or (13). In the analysis of the summation CFAR detector, it is found that it is more appropriate to focus on the quantity, $T/(LN\sigma^2)$, which will be called the normalized detection threshold in this paper. This can be loosely interpreted as the detection threshold for one FFT bin computed from a single data block.

IV. Lower and Upper Bounds for the Detection Threshold T

In practice, the computation of T/σ^2 involves the numerical solution of one of equations (5), (11) or (13) using numerical procedures such as the Newton-Ralphson algorithm. To avoid potential numerical problems and ensure fast convergence, a good lower or upper bound for T is required to initialize the numerical procedures. This section presents theoretical lower and upper bounds for T. The cases for overlapped and non-overlapped signal data are treated separately, with full proofs provided in the appendix.

A. Overlapped Input Data

Theorem 4. Assume $L \geq 2$, $0 < \gamma \leq 1/2$, and let $\lambda_1 > \lambda_2 > \cdots > \lambda_{LN}$ be the LN distinct positive eigenvalues of the $LN \times LN$ Hermitian matrix $\mathbf{H}$ defined by (6). For a given threshold $T > 0$, let the probability of false alarm be denoted by P_{fa}. Let $T_1(P_{fa}) = -\lambda_1 \sigma^2 \ln P_{fa}$ and denote the solution of the following equation in x by $T_n(P_{fa})$:

$$P_{fa} = \sum_{m=1}^{n} \frac{\lambda_m^{n-1}}{\prod\limits_{1 \leq l \leq n, l \neq m} (\lambda_m - \lambda_l)} e^{-\frac{x}{\sigma^2 \lambda_m}}, \quad 2 \leq n \leq LN. \tag{15}$$

Then

$$T_1(P_{fa}) \leq T_2(P_{fa}) \leq \cdots \leq T_l(P_{fa}) \leq T_{l+1}(P_{fa}) \leq \cdots \leq T_{LN}(P_{fa}) = T. \tag{16}$$

For any positive integer $n \geq 1$, let the unique solution of the following equation in z be denoted by $B_n(P_{fa})$:

$$e^{-z}\left(1 + z + \frac{z^2}{2!} + \cdots + \frac{z^{n-1}}{(n-1)!}\right) = P_{fa}. \tag{17}$$

Then for $0 < P_{fa} \leq 2/e$,

$$B_n(P_{fa}) \leq n + \frac{\sqrt{2\pi n}}{2} \ln \left[\frac{2}{eP_{fa}} \right], \tag{18}$$

and for $0 < P_{fa} < 1$,

$$T \leq \lambda_1 \sigma^2 B_{LN}(P_{fa}). \tag{19}$$

B. Non-Overlapped Input Data

Theorem 5. Assume $L \geq 1$, $N \geq 2$, $\gamma = 0$ and let $\mu_1 > \mu_2 > \cdots > \mu_N$ be the N distinct positive eigenvalues of the Hermitian matrix $\mathbf{A}$ defined by (7). We have

$$\mu_1\sigma^2 B_L(P_{fa}) \leq T \leq \mu_1\sigma^2 B_{LN}(P_{fa}). \quad (20)$$

Moreover, if T_1 and T_2 are respectively the solutions of the equations (21) and (22) in x:

$$\sum_{m=1}^{N} \frac{\mu_m^{N-1}}{\prod_{1\leq l\leq N, l\neq m}(\mu_m - \mu_l)} e^{-\frac{x}{\sigma^2\mu_m}} = (P_{fa})^{\frac{1}{L}}, \quad (21)$$

$$\sum_{m=1}^{N} \frac{\mu_m^{N-1}}{\prod_{1\leq l\leq N, l\neq m}(\mu_m - \mu_l)} e^{-\frac{x}{\sigma^2\mu_m}} = 1 - (1 - P_{fa})^{\frac{1}{L}}, \quad (22)$$

then

$$LT_1 \leq T \leq LT_2, \quad (23)$$

with

$$T_2 \leq \mu_1\sigma^2 B_N\left(1 - (1 - P_{fa})^{\frac{1}{L}}\right). \quad (24)$$

V. Numerical Computation of the Normalized Detection Threshold for the Summation CFAR Detector

Using (5), (11) and (13) and in conjuction with the lower and upper bounds derived in the previous section, the normalized detection threshold $T/(LN\sigma^2)$ can be computed numerically. It has been observed that in general, equation (5) is numerically more unstable and hence more difficult to apply in practice than equation (11). To avoid numerical problems with (5), the transformation $p = e^{\frac{-T}{\sigma^2\lambda_1}}$ has proved to be very useful. Define the function $g(p)$ by:

$$g(p) = \sum_{l=1}^{LN} \frac{\lambda_l^{LN-1}}{\prod\limits_{1\leq m\leq LN, m\neq l}(\lambda_l - \lambda_m)} p^{\frac{\lambda_1}{\lambda_l}}, \quad 0 < p < 1. \quad (25)$$

Then the formula (5) can be rewritten as:

$$g\left(e^{\frac{-T}{\sigma^2\lambda_1}}\right) = P_{fa}. \quad (26)$$

The detection threshold T can be obtained by first solving the equation $g(p) = P_{fa}$ for p and then using the relationship $T = \lambda_1\sigma^2\ln(1/p)$. It can be shown that $g(p)$ is concave on the interval $(0, 1)$, as is demonstrated by Figure 2. For comparison, P_{fa} is plotted as a function of T/σ^2 using (5) in Figure 3. The concavity of $g(p)$ is a very desirable property to have in the numerical computation of $T/(LN\sigma^2)$; it guarantees the fast global quadratic convergence of the Newton-Ralphson algorithm provided that the functions $g(p)$ and $g'(p)$ can be reliably computed. The proof of the concavity of $g(p)$ is rather non-trivial and will be given in the appendix:

Theorem 6. The function $g(p)$ is strictly increasing and concave on the interval $(0, 1)$. Specifically, $g'(p) > 0$ and $g''(p) < 0$ for $0 < p < 1$.

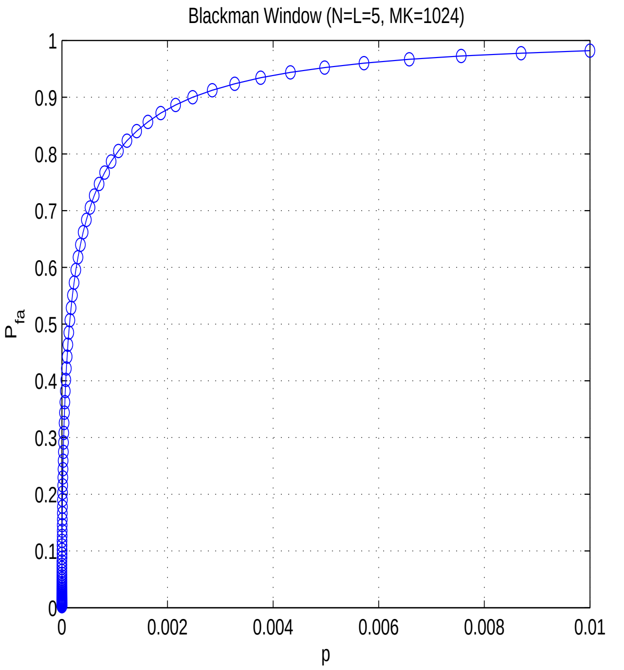

Fig. 2. The probability of false alarm, P_{fa}, as a function of $p = e^{-T/(\sigma^2\lambda_1)}$, $N = L = 5$, $MK = 1024$, $\gamma = 0.5$, Blackman window.

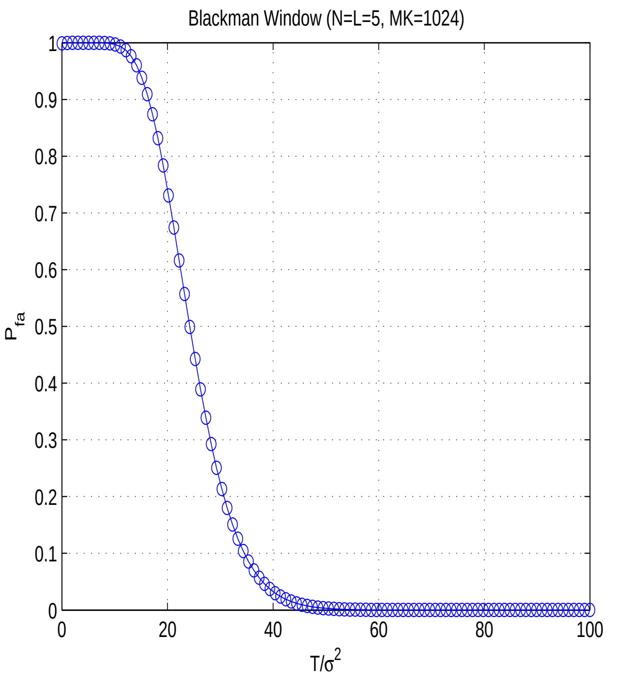

Fig. 3. The probability of false alarm, P_{fa}, as a function of T/σ^2, $N = L = 5$, $MK = 1024$, $\gamma = 0.5$, Blackman window.

For LN not too large, we have successfully computed the normalized detection threshold, $T/(LN\sigma^2)$, for the L-block summation CFAR detector. Typical results are plotted in Figures 4-5 for $\gamma = 0$. Note that the windows used to produce these results have been normalized in the sense that $\sum_{l=0}^{MK-1} w_l^2 = 1$. The results for $\gamma = 1/2$ are very similar, but are not shown due to space limitations.

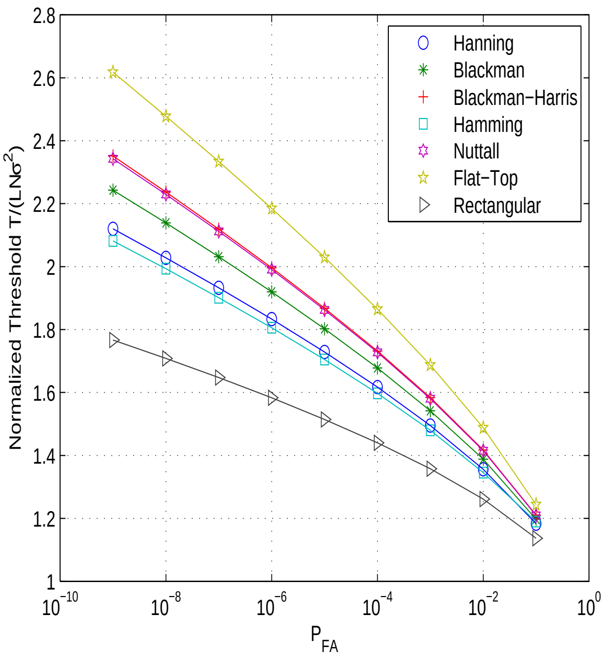

Fig. 4. The normalized detection threshold, $T/(LN\sigma^2)$, as a function of the probability of false alarm P_{fa}, $N = 5$, $MK = 1024$, $L = 18$, $\gamma = 0$. The window functions are normalized.

VI. Accurate Approximations to the Detection Threshold

For very large LN, there are serious numerical difficulties in computing the normalized detection threshold $T/(LN\sigma^2)$ using (5) or (11). In such cases accurate approximations to $T/(LN\sigma^2)$ are necessary. As will be demonstrated in the appendix in the proofs of **Theorems 1-3**, the detection decision statistic, $\sum_{l=0}^{L-1} z_{l,k}$, of the k-th channel is distributed as a weighted sum of chi-square random variables. More specifically,

$$\sum_{l=0}^{L-1} z_{l,k} \sim \begin{cases} \sum_{l=1}^{LN} \frac{\sigma^2 \lambda_l}{2} \chi_2^2(l), & 0 < \gamma \le 1/2, \\ \sum_{m=1}^{N} \frac{\sigma^2 \mu_m}{2} \chi_{2L}^2(m), & \gamma = 0, \end{cases} \tag{27}$$

where λ_l, $1 \le l \le LN$, are the LN distinct positive eigenvalues of the Hermitian matrix $\mathbf{H}$ defined by (6) and μ_m, $1 \le m \le N$, are the N distinct positive eigenvalues of the Hermitian matrix $\mathbf{A}$ defined by (7). As pointed out in [15], one can approximate a weighted sum of chi-square random variables by a random variable of the form $c + d\chi_h^2$ where c and d are constants and χ_h is a chi-square random variable with h degrees of freedom.

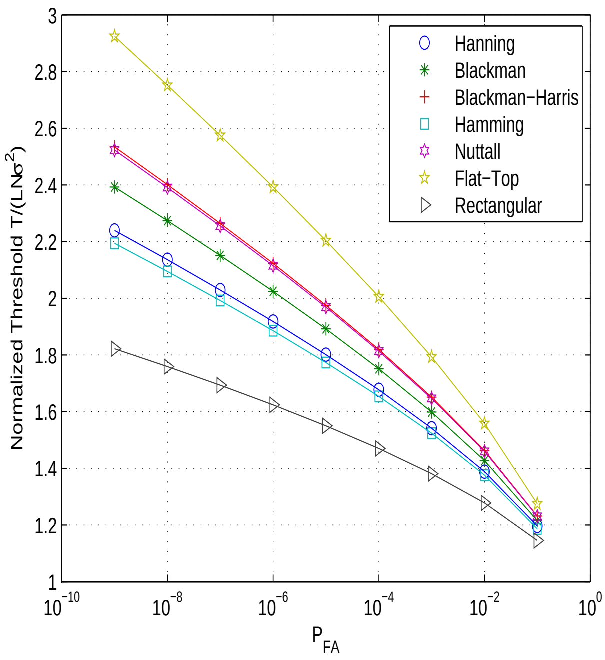

Fig. 5. The normalized detection threshold, $T/(LN\sigma^2)$, as a function of the probability of false alarm P_{fa}, with $N = 8$, $MK = 1024$, $L = 10$, $\gamma = 0$. The window functions are normalized.

This implies that the normalized detection threshold, $T/(LN\sigma^2)$, can be approximately computed via a chi-square distribution. Using this approach, the following two theorems have been derived for normalized windows (i.e., $\sum_{l=0}^{MK-1} w_l^2 = 1$).

Theorem 7. Assume $L \ge 2$ and $\gamma = 0$. For a given P_{fa}, T is approximately computed by:

$$T \approx \sigma^2 \left[L \left[N - \frac{\left(\sum_{m=1}^{N} \mu_m^2\right)^2}{\sum_{m=1}^{N} \mu_m^3} \right] + B_h(P_{fa}) \frac{\sum_{m=1}^{N} \mu_m^3}{\sum_{m=1}^{N} \mu_m^2} \right],$$

$$h = \left\lfloor L \frac{\left(\sum_{m=1}^{N} \mu_m^2\right)^3}{\left(\sum_{m=1}^{N} \mu_m^3\right)^2} \right\rfloor, \tag{28}$$

where μ_m, $1 \le m \le N$, are the N distinct positive eigenvalues of $\mathbf{A}$ defined by (7) and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Theorem 8. Assume $L \ge 2$ and $0 < \gamma \le 1/2$. For a given P_{fa}, T is approximately computed by:

$$T \approx \sigma^2 \left[LN - \frac{\left(\sum_{l=1}^{LN} \lambda_l^2\right)^2}{\sum_{l=1}^{LN} \lambda_l^3} + B_h(P_{fa}) \frac{\sum_{l=1}^{LN} \lambda_l^3}{\sum_{l=1}^{LN} \lambda_l^2} \right],$$

$$h = \left\lfloor \frac{\left(\sum_{l=1}^{LN} \lambda_l^2\right)^3}{\left(\sum_{l=1}^{LN} \lambda_l^3\right)^2} \right\rfloor, \tag{29}$$

where λ_l, $1 \le l \le LN$, are the LN distinct positive eigenvalues of $\mathbf{H}$ defined by (6).

The excellent accuracy of approximations (28) and (29) is apparent in Figure 6. This figure shows the true normalized detection thresholds for the cases of $\gamma = 0$ and $\gamma = 1/2$ and the corresponding approximate normalized detection thresholds plotted as a function of P_{fa} for a typical example. In this figure's legend, "NO" and "O" denote non-overlapped ($\gamma = 0$) and overlapped ($\gamma = 1/2$), respectively. It follows that, the approximations (28) and (29) can be used to avoid numerical problems in the direct computation of the normalized detection threshold.

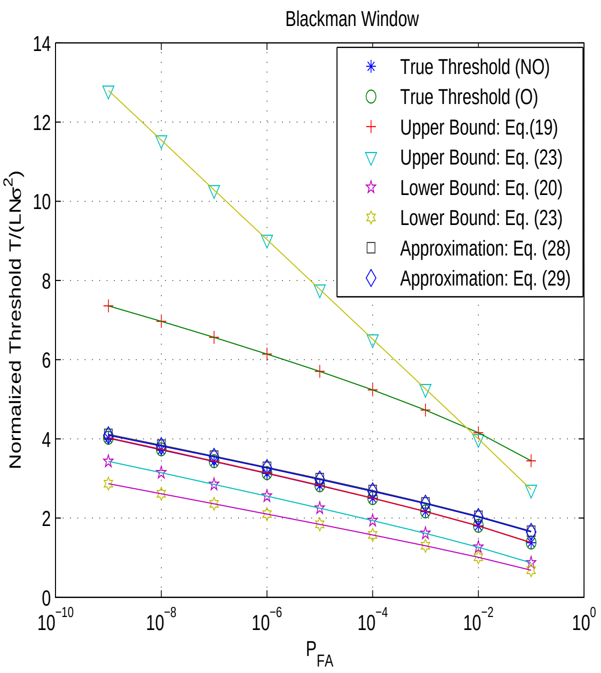

Fig. 6. Upper and lower bounds and true and approximate normalized detection thresholds as a function of the probability of false alarm P_{fa}, $N = L = 5$, $MK = 1024$, $\gamma = 0$ or $\gamma = 1/2$, Blackman window.

VII. Conclusions

The problem of computing the normalized detection threshold for the FFT filter bank-based summation CFAR detector has been comprehensively treated. Formulas for computing the probability of false alarm, P_{fa}, as a function of the detection threshold, T, were derived from first principles. In addition, accurate approximations for T that avoid potential numerical problems are presented. These results are very useful for the simulation and implementation of the FFT filter bank-based summation CFAR detector.

VIII. Acknowledgements

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Appendix

Proof of Theorems 1-3. Assume $0 \le \gamma \le 1/2$ and let $I_k = kK + \frac{K-N}{2}$. For $0 \le l \le L-1$, we have

$$\begin{aligned} z_{l,k} &= \sum_{m=0}^{N-1} |s_{l,I_k+m}|^2 = \sum_{m=0}^{N-1} s_{l,I_k+m}(s_{l,I_k+m})^* \\ &= \mathbf{Z}_{l,k}^H \mathbf{Z}_{l,k}, \end{aligned} \tag{30}$$

$$\begin{aligned} \mathbf{Z}_{l,k} &= [s_{l,I_k},\ s_{l,I_k+1},\ \cdots,\ s_{l,I_k+N-1}]^t \\ &= \mathbf{F}_k \mathbf{W} \mathbf{R}_l, \end{aligned} \tag{31}$$

where in (31), $\mathbf{R}_l$ is the l-th sample vector defined by (1), $\mathbf{W}$ is the $MK \times MK$ diagonal matrix with its diagonal elements defined by $\mathbf{w}$:

$$\mathbf{W} = \begin{bmatrix} w_0 & 0 & \cdots & \cdots & 0 \\ 0 & w_1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & w_{MK-1} \end{bmatrix}, \tag{32}$$

and $\mathbf{F}_k$ is the $N \times MK$ matrix consisting of N rows of the $MK \times MK$ inverse discrete Fourier transform matrix $\mathbf{F}$ with row indices $I_k, I_k+1, \cdots, I_k+m, \cdots, I_k+N-1$:

$$\mathbf{F}_k = \tag{33}$$
$$\begin{bmatrix} 1 & \alpha^{I_k} & \cdots & \alpha^{(MK-1)I_k} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \alpha^{I_k+m} & \cdots & \alpha^{(MK-1)(I_k+m)} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \alpha^{I_k+N-1} & \cdots & \alpha^{(MK-1)(I_k+N-1)} \end{bmatrix},$$

where $\alpha = \exp \frac{2\pi j}{MK}$. Hence

$$\sum_{l=0}^{L-1} z_{l,k} = \sum_{l=0}^{L-1} \mathbf{Z}_{l,k}^H \mathbf{Z}_{l,k} = \mathbf{Z}^H \mathbf{Z}, \tag{34}$$

where $\mathbf{Z}$ is the length-LN zero mean Gaussian random vector defined by:

$$\mathbf{Z} = \left[\mathbf{Z}_{0,k}^t, \mathbf{Z}_{1,k}^t, \cdots, \mathbf{Z}_{L-1,k}^t\right]^t. \tag{35}$$

From the identity (4a) of [16], the characteristic function of the quadratic form $\sum_{l=0}^{L-1} z_{l,k} = \mathbf{Z}^H\mathbf{Z}$, denoted by $\phi(t)$, can be shown to be given by:

$$\phi(t) = \frac{1}{\det\left|\mathbf{I} - jtE\left(\mathbf{Z}\mathbf{Z}^H\right)\right|}, \tag{36}$$

where $\mathbf{I}$ denotes the identity matrix of dimensions $LN \times LN$. We shall derive the equations (5), (11) and (13) using the characteristic function $\phi(t)$. First, it is necessary to compute the Hermitian matrix $E\left(\mathbf{Z}\mathbf{Z}^H\right)$ explicitly. In fact, $E\left(\mathbf{ZZ}^H\right)$ is given by:

$$E\left(\mathbf{ZZ}^H\right) = E\left(\begin{bmatrix} \mathbf{Z}_{0,k} \\ \mathbf{Z}_{1,k} \\ \cdots \\ \mathbf{Z}_{p,k} \\ \cdots \\ \mathbf{Z}_{L-1,k} \end{bmatrix} \begin{bmatrix} \mathbf{Z}_{0,k} \\ \mathbf{Z}_{1,k} \\ \cdots \\ \mathbf{Z}_{p,k} \\ \cdots \\ \mathbf{Z}_{L-1,k} \end{bmatrix}^H\right) =$$
$$\begin{bmatrix} E\left(\mathbf{Z}_{0,k}\mathbf{Z}_{0,k}^H\right) & \cdots & E\left(\mathbf{Z}_{0,k}\mathbf{Z}_{L-1,k}^H\right) \\ \cdots & \cdots & \cdots \\ E\left(\mathbf{Z}_{p,k}\mathbf{Z}_{0,k}^H\right) & \cdots & E\left(\mathbf{Z}_{p,k}\mathbf{Z}_{L-1,k}^H\right) \\ \cdots & \cdots & \cdots \\ E\left(\mathbf{Z}_{L-1,k}\mathbf{Z}_{0,k}^H\right) & \cdots & E\left(\mathbf{Z}_{L-1,k}\mathbf{Z}_{L-1,k}^H\right) \end{bmatrix}. \tag{37}$$

For any $l,\ m,\ \ 0 \le l \le m \le L-1$,

$$\begin{aligned} & E\left(\mathbf{Z}_{l,k}\mathbf{Z}_{m,k}^H\right) \\ &= E\left(\left(\mathbf{F}_k\mathbf{W}\mathbf{R}_l\right)\left(\mathbf{F}_k\mathbf{W}\mathbf{R}_m\right)^H\right) \\ &= E\left(\mathbf{F}_k\mathbf{W}\mathbf{R}_l\mathbf{R}_m^H\mathbf{W}^H\mathbf{F}_k^H\right) \\ &= \mathbf{F}_k\mathbf{W}E\left(\mathbf{R}_l\mathbf{R}_m^H\right)\mathbf{W}\mathbf{F}_k^H, \end{aligned} \tag{38}$$

where it can be verified that

$$E\left(\mathbf{R}_l\mathbf{R}_m^H\right) = \sigma^2 \times \tag{39}$$
$$\begin{bmatrix} \delta_J & \cdots & \delta_{J-m} & \cdots & \delta_{J-MK+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \delta_{J+n-1} & \cdots & \delta_{J+n-1-m} & \cdots & \delta_{J-MK+n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \delta_{J+MK-1} & \cdots & \delta_{J+MK-1-m} & \cdots & \delta_J \end{bmatrix},$$

with $J = (m-l)(1-\gamma)MK$.

Here in (39), $\delta_q = 0$, if $q \neq 0$ and $\delta_q = 1$ if $q = 0$. It is clear that the matrix $E\left(\mathbf{R}_l\mathbf{R}_m^H\right)$ is shift invariant in the sense that it depends only on the value of $m-l$. Let

$$\begin{aligned} \mathbf{A}_p &= E\left(\mathbf{Z}_{l,k}\mathbf{Z}_{m,k}^H\right) \\ &= \mathbf{F}_k\mathbf{W}E\left(\mathbf{R}_l\mathbf{R}_m^H\right)\mathbf{W}\mathbf{F}_k^H, \\ & \quad p = m-l,\ m \ge l. \end{aligned} \tag{40}$$

The covariance matrix $E\left(\mathbf{ZZ}^H\right)$ can then be put in the following block Toeplitz form:

$$E\left(\mathbf{ZZ}^H\right) = \tag{41}$$
$$\begin{bmatrix} \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_{L-2} & \mathbf{A}_{L-1} \\ \mathbf{A}_1^H & \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_{L-2} \\ \mathbf{A}_2^H & \mathbf{A}_1^H & \mathbf{A}_0 & \mathbf{A}_1 & \cdots & \mathbf{A}_{L-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{A}_{L-3}^H & \cdots & \mathbf{A}_1^H & \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_{L-2}^H & \cdots & \mathbf{A}_2^H & \mathbf{A}_1^H & \mathbf{A}_0 & \mathbf{A}_1 \\ \mathbf{A}_{L-1}^H & \mathbf{A}_{L-2}^H & \cdots & \mathbf{A}_2^H & \mathbf{A}_1^H & \mathbf{A}_0 \end{bmatrix}.$$

If $m-l \ge 2$, then $|(m-l)(1-\gamma)MK| \ge 2(1-\gamma)MK \ge 2(1-1/2)MK = MK$ and hence entries in $E\left(\mathbf{R}_l\mathbf{R}_m^H\right)$ are all equal to zero. Thus $\mathbf{A}_p = \mathbf{0}$ if $p \ge 2$ and $\mathbf{H} =$ $E\left(\mathbf{ZZ}^H\right)$ simplifies to the following tridiagonal $L \times L$ block matrix:

$$E\left(\mathbf{ZZ}^H\right) = \tag{42}$$
$$\begin{bmatrix} \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_1^H & \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_1^H & \mathbf{A}_0 & \mathbf{A}_1 \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{A}_1^H & \mathbf{A}_0 \end{bmatrix},$$

where $\mathbf{A}_0 = E\left(\mathbf{Z}_{0,k}\mathbf{Z}_{0,k}^H\right)$ and $\mathbf{A}_1 = E\left(\mathbf{Z}_{0,k}\mathbf{Z}_{1,k}^H\right)$ are of dimensions $N \times N$.

We now consider the two separate cases $\gamma = 0$ and $0 < \gamma \le 1/2$.

1) $\gamma = 0$. In this case, $\mathbf{A}_1 = 0$ and $E\left(\mathbf{ZZ}^H\right)$ becomes the following diagonal $L \times L$ block matrix

$$E\left(\mathbf{ZZ}^H\right) = \tag{43}$$
$$\begin{bmatrix} \mathbf{A}_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{A}_0 & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_0 \end{bmatrix},$$

where

$$\begin{aligned} \mathbf{A}_0 &= E\left(\mathbf{Z}_{0,k}\mathbf{Z}_{0,k}^H\right) \\ &= \mathbf{F}_k\mathbf{W}E\left(\mathbf{R}_0\mathbf{R}_0^H\right)\mathbf{W}\mathbf{F}_k^H \\ &= \sigma^2\mathbf{F}_k\mathbf{W}^2\mathbf{F}_k^H. \end{aligned} \tag{44}$$

With $I_k = kK + \frac{K-N}{2}$, $J = MK$ and $\alpha = \exp\frac{2\pi j}{MK}$, we obtain:

$$\mathbf{A}_0 = \sigma^2\mathbf{F}_k\mathbf{W}^2\mathbf{F}_k^H = \sigma^2 \times$$
$$\begin{bmatrix} w_0^2 & w_1^2\alpha^{I_k} & \cdots & w_{MK-1}^2\alpha^{(MK-1)I_k} \\ \cdots & \cdots & \cdots & \cdots \\ w_0^2 & w_1^2\alpha^{I_k+l} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ w_0^2 & w_1^2\alpha^{I_k+N-1} & \cdots & \cdots \end{bmatrix}$$
$$\times \begin{bmatrix} 1 & \cdots & 1 \\ \cdots & \cdots & \cdots \\ \alpha^{-lI_k} & \cdots & \alpha^{-l(I_k+N-1)} \\ \cdots & \cdots & \cdots \\ \alpha^{-(MK-1)I_k} & \cdots & \alpha^{-(MK-1)(I_k+N-1)} \end{bmatrix}$$
$$= \sigma^2 \begin{bmatrix} \tau_{11} & \tau_{12} & \cdots & \tau_{1q} & \cdots & \tau_{1N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \tau_{p1} & \tau_{p2} & \cdots & \tau_{pq} & \cdots & \tau_{pN} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \tau_{N1} & \tau_{N2} & \cdots & \tau_{Nq} & \cdots & \tau_{NN} \end{bmatrix}$$
$$= \sigma^2\mathbf{A}, \tag{45}$$

where, for $1 \le p \le q \le N$,

$$\begin{aligned} \tau_{pq} &= \sum_{l=0}^{MK-1} w_l^2\alpha^{l(p-q)} \\ &= \sum_{l=0}^{MK-1} w_l^2 \exp\frac{2\pi jl(p-q)}{MK}, \end{aligned} \tag{46}$$

and $\mathbf{A}$ is defined by (7).

2) $0 < \gamma \le 1/2$. In this case $\mathbf{A}_0$ is computed by (45) and it remains to compute $\mathbf{A}_1$. In fact,

$$E\left(\mathbf{R}_0\mathbf{R}_1^H\right) = \sigma^2 \begin{bmatrix} \mathbf{0}_{\gamma MK\times(1-\gamma)MK} & \mathbf{I}_{\gamma MK\times\gamma MK} \\ \mathbf{0}_{(1-\gamma)MK\times(1-\gamma)MK} & \mathbf{0}_{(1-\gamma)MK\times\gamma MK} \end{bmatrix}, \tag{47}$$

where $\mathbf{0}_{\gamma MK\times(1-\gamma)MK}$, $\mathbf{0}_{(1-\gamma)MK\times(1-\gamma)MK}$ and $\mathbf{0}_{(1-\gamma)MK\times\gamma MK}$ are zero matrices of dimensions $\gamma MK\times(1-\gamma)MK$, $(1-\gamma)MK\times(1-\gamma)MK$ and $(1-\gamma)MK\times\gamma MK$ respectively and $\mathbf{I}_{\gamma MK\times\gamma MK}$ is the identity matrix of dimensions $\gamma MK\times\gamma MK$. Let

$$\mathbf{W}_1 = \begin{bmatrix} w_0 & 0 & \cdots & 0 \\ 0 & w_1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & w_{\gamma MK-1} \end{bmatrix}, \tag{48}$$

$$\mathbf{W}_2 = \begin{bmatrix} w_0 & 0 & \cdots & 0 \\ 0 & w_1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & w_{(1-\gamma)MK-1} \end{bmatrix}, \tag{49}$$

$$\mathbf{W}_3 = \begin{bmatrix} w_{\gamma MK} & 0 & \cdots & 0 & 0 \\ 0 & w_{\gamma MK+1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & w_{MK-1} \end{bmatrix}, \tag{50}$$

$$\mathbf{W}_4 = \begin{bmatrix} w_{(1-\gamma)MK} & 0 & \cdots & 0 \\ 0 & w_{(1-\gamma)MK+1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & w_{MK-1} \end{bmatrix}. \tag{51}$$

We obtain

$$\begin{aligned} \mathbf{A}_1 &= E\left(\mathbf{Z}_{0,k}\mathbf{Z}_{1,k}^H\right) \\ &= \mathbf{F}_k\mathbf{W}E\left(\mathbf{R}_0\mathbf{R}_1^H\right)\mathbf{W}\mathbf{F}_k^H \\ &= \sigma^2\mathbf{F}_k \begin{bmatrix} \mathbf{W}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_3 \end{bmatrix} \times \begin{bmatrix} \mathbf{0}_{\gamma MK\times(1-\gamma)MK} & \mathbf{I}_{\gamma MK\times\gamma MK} \\ \mathbf{0}_{(1-\gamma)MK\times(1-\gamma)MK} & \mathbf{0}_{(1-\gamma)MK\times\gamma MK} \end{bmatrix} \\ &\quad \times \begin{bmatrix} \mathbf{W}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_4 \end{bmatrix} \mathbf{F}_k^H \\ &= \sigma^2\mathbf{F}_k \times \begin{bmatrix} \mathbf{0}_{\gamma MK\times(1-\gamma)MK} & \mathbf{W}_1\mathbf{W}_4 \\ \mathbf{0}_{(1-\gamma)MK\times(1-\gamma)MK} & \mathbf{0}_{(1-\gamma)MK\times\gamma MK} \end{bmatrix} \\ &\quad \times\mathbf{F}_k^H. \end{aligned} \tag{52}$$

Let $\alpha = \exp(\frac{2\pi j}{MK})$, $J = (1-\gamma)MK$ and set

$$\mathbf{F}_k^0 = \begin{bmatrix} 1 & \alpha^{I_k} & \cdots & \alpha^{(\gamma MK-1)I_k} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \alpha^{I_k+p-1} & \cdots & \alpha^{(\gamma MK-1)(I_k+p-1)} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \alpha^{I_k+N-1} & \cdots & \alpha^{(\gamma MK-1)(I_k+N-1)} \end{bmatrix}, \tag{53}$$

and

$$\mathbf{F}_k^1 = \begin{bmatrix} \alpha^{JI_k} & \cdots & \alpha^{(MK-1)I_k} \\ \cdots & \cdots & \cdots \\ \alpha^{J(I_k+p-1)} & \cdots & \alpha^{(MK-1)(I_k+p-1)} \\ \cdots & \cdots & \cdots \\ \alpha^{J(I_k+N-1)} & \cdots & \alpha^{(MK-1)(I_k+N-1)} \end{bmatrix}. \tag{54}$$

Then $\mathbf{A}_1$ can be further simplified to yield:

$$\begin{aligned} \mathbf{A}_1 &= E\left(\mathbf{Z}_{0,k}\mathbf{Z}_{1,k}^H\right) \\ &= \sigma^2\mathbf{F}_k \begin{bmatrix} \mathbf{0} & \mathbf{W}_1\mathbf{W}_4 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{F}_k^H \\ &= \sigma^2\, \mathbf{F}_k^0\, \mathbf{W}_1\mathbf{W}_4\, \left(\mathbf{F}_k^1\right)^H \\ &= \sigma^2\mathbf{C}, \end{aligned} \tag{55}$$

where

$$\mathbf{C} = \mathbf{F}_k^0\, \mathbf{W}_1\mathbf{W}_4\, \left(\mathbf{F}_k^1\right)^H = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1q} & \cdots & \gamma_{1N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{p1} & \gamma_{p2} & \cdots & \gamma_{pq} & \cdots & \gamma_{pN} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{N1} & \gamma_{N2} & \cdots & \gamma_{Nq} & \cdots & \gamma_{NN} \end{bmatrix}, \tag{56}$$

with

$$\begin{aligned} \gamma_{pq} &= \sum_{l=0}^{\gamma MK-1} \alpha^{l(I_k+p-1)} \times w_l w_{l+(1-\gamma)MK}\alpha^{-(l+(1-\gamma)MK)(I_k+q-1)} \\ &= e^{-2\pi j(1-\gamma)(I_k+q-1)} \times \sum_{l=0}^{\gamma MK-1} w_l w_{l+(1-\gamma)MK} \exp\frac{2\pi jl(p-q)}{MK}. \end{aligned} \tag{57}$$

Summarizing the preceding calculations, we see that $E\left(\mathbf{Z}\mathbf{Z}^H\right) = \sigma^2\mathbf{H}$ where $\mathbf{H}$ is defined by (6) with $\mathbf{B} = \mathbf{0}$ if $\gamma = 0$.

Theorems 1-3 can now be proved using the following identity:

$$\frac{1}{2\pi}\int_T^{+\infty}\left[\int_{-\infty}^{+\infty}\frac{e^{-2\pi jtx}}{(1-jt\lambda)^p}dt\right]dx = \sum_{s=0}^{p-1}\frac{\left(\frac{T}{\lambda}\right)^s}{s!}\, e^{-\frac{T}{\lambda}}, \tag{58}$$

where $T > 0$, $\lambda > 0$ and p is a positive integer. This identity can be proved using an elaborate but standard contour integral argument from the theory of functions of one complex variable. Details are omitted due to space constraints.

We first prove **Theorem 1**. Assume $0 < \gamma \le 1/2$. Let the distinct positive eigenvalues of $\mathbf{H}$ defined by (6) be arranged in decreasing order and denoted by $\lambda_1 > \lambda_2 > \cdots > \lambda_{LN} > 0$. From (36) it follows that the characteristic function $\phi(t)$ of $\sum_{l=0}^{L-1} z_{l,k} = \mathbf{Z}^H\mathbf{Z}$ is given by:

$$\begin{aligned}\phi(t) &= \frac{1}{\det|\mathbf{I}-jtE(\mathbf{ZZ}^H)|} \quad (59)\\ &= \frac{1}{\det|\mathbf{I}-jt\sigma^2\mathbf{H}|} = \frac{1}{\prod_{l=1}^{LN}(1-jt\sigma^2\lambda_l)}\\ &= \sum_{l=1}^{LN}\frac{\lambda_l^{LN-1}}{\prod\limits_{1\le m\le LN, m\ne l}(\lambda_l-\lambda_m)}\frac{1}{1-jt\sigma^2\lambda_l}.\end{aligned}$$

This implies that $\sum_{l=0}^{L-1} z_{l,k} = \mathbf{Z}^H\mathbf{Z}$ is distributed as the following weighted sum of chi-square random variables:

$$\sum_{l=0}^{L-1} z_{l,k} = \mathbf{Z}^H\mathbf{Z} \sim \sum_{l=1}^{LN}\frac{\sigma^2\lambda_l}{2}\chi_2^2(l), \quad (60)$$

where $\chi_2^2(l), 1 \le l \le LN$, are independent and identically distributed chi-square random variables, each of two degrees of freedom, and $\sim$ means having identical probability density functions. Note that here we used the fact that the characteristic function of $\frac{\sigma^2\lambda_l}{2}\chi_2^2(l)$ is given by $\frac{1}{1-jt\sigma^2\lambda_l}$. Let $p(x)$ denote the probability density function of the detection decision statistic $\sum_{l=0}^{L-1} z_{l,k} = \mathbf{Z}^H\mathbf{Z}$. We have

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\phi(t)e^{-2\pi jtx}dt, \; x > 0, \quad (61)$$

and for a given $T > 0$, P_{fa} is then computed by:

$$\begin{aligned}P_{fa} &= \left\{\sum_{l=0}^{L-1} z_{l,k} > T\right\} = \int_T^{+\infty} p(x)dx\\ &= \int_T^{+\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\phi(t)e^{-2\pi jtx}dt\right]dx\\ &= \sum_{l=1}^{LN}\frac{\lambda_l^{LN-1}}{\prod\limits_{1\le m\le LN, m\ne l}(\lambda_l-\lambda_m)}\times\\ &\quad \frac{1}{2\pi}\int_T^{+\infty}\left[\int_{-\infty}^{+\infty}\frac{e^{-2\pi jtx}}{1-jt\sigma^2\lambda_l}dt\right]dx\\ &= \sum_{l=1}^{LN}\frac{\lambda_l^{LN-1}}{\prod\limits_{1\le m\le LN, m\ne l}(\lambda_l-\lambda_m)}e^{-\frac{T}{\sigma^2\lambda_l}}, \quad (62)\end{aligned}$$

where the formula (58) with $p = 1$ was used. This completes the proof of Theorem 1.

To prove **Theorem 2**, let the distinct positive eigenvalues of the positive definite Hermitian matrix $\mathbf{A}$ defined by (7) be arranged in decreasing order and denoted by $\mu_1 > \mu_2 > \cdots > \mu_N$. It follows from (36) that the characteristic function $\phi(t)$ of $\sum_{l=0}^{L-1} z_{l,k} = \mathbf{Z}^H\mathbf{Z}$ is given by:

$$\begin{aligned}\phi(t) &= \frac{1}{\det|\mathbf{I}-jtE(\mathbf{ZZ}^H)|}\\ &= \frac{1}{\det|\mathbf{I}-jt\sigma^2\mathbf{H}|}\\ &= \frac{1}{\prod_{m=1}^{N}(1-jt\sigma^2\mu_m)^L}\\ &= \sum_{m=1}^{N}\sum_{p=1}^{L}A_{mp}\frac{1}{(1-jt\sigma^2\mu_m)^p}, \quad (63)\end{aligned}$$

where A_{mp}, $1 \le m \le N$, $1 \le p \le L$, are defined by (12). The routine but tedious derivations of the fractional decomposition of the rational function $\frac{1}{\prod_{m=1}^{N}(1-jt\sigma^2\mu_m)^L}$ are omitted due to space constraints. It then follows that the detection decision statistic $\sum_{l=0}^{L-1} z_{l,k}$ is distributed as the following weighted sum of chi-square random variables:

$$\sum_{l=0}^{L-1} z_{l,k} = \mathbf{Z}^H\mathbf{Z} \sim \sum_{m=1}^{N}\frac{\sigma^2\mu_m}{2}\chi_{2L}^2(m), \quad (64)$$

where $\chi_{2L}^2(m), 1 \le m \le N$, are independent and identically distributed chi-square random variables each of $2L$ degrees of freedom. For a given $T > 0$, P_{fa} is given by:

$$\begin{aligned}P_{fa} &= \left\{\sum_{l=0}^{L-1} z_{l,k} > T\right\} = \int_T^{+\infty} p(x)dx \quad (65)\\ &= \int_T^{+\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\phi(t)e^{-2\pi jtx}dt\right]dx\\ &= \sum_{m=1}^{N}\sum_{p=1}^{L}\frac{A_{mp}}{2\pi}\int_T^{+\infty}\left[\int_{-\infty}^{+\infty}\frac{e^{-2\pi jtx}}{(1-jt\sigma^2\mu_m)^p}dt\right]dx\\ &= \sum_{m=1}^{N}\sum_{p=1}^{L}A_{mp}\sum_{s=0}^{p-1}\frac{\left(\frac{T}{\sigma^2\mu_m}\right)^s}{s!}e^{-\frac{T}{\sigma^2\mu_m}}.\end{aligned}$$

This completes the proof of **Theorem 2**.

It remains to prove **Theorem 3**. Since $N = 1$, the covariance matrix $E(\mathbf{ZZ}^H)$ reduces to a $L \times L$ diagonal matrix with the diagonal elements all equal to $\sigma^2\sum_{l=0}^{MK} w_l^2$. It follows from (36) that the characteristic function $\phi(t)$ of $\sum_{l=0}^{L-1} z_{l,k} = \mathbf{Z}^H\mathbf{Z}$ is given by:

$$\begin{aligned}\phi(t) &= \frac{1}{\det|\mathbf{I}-jtE(\mathbf{ZZ}^H)|}\\ &= \frac{1}{\left(1-jt\sigma^2\sum_{l=0}^{MK} w_l^2\right)^L}. \quad (66)\end{aligned}$$

This implies that $\sum_{l=0}^{L-1} z_{l,k} = \mathbf{Z}^H\mathbf{Z}$ is distributed as a weighted chi-square random variable:

$$\sum_{l=0}^{L-1} z_{l,k} = \mathbf{Z}^H\mathbf{Z} \sim \frac{\sigma^2\sum_{l=0}^{MK-1} w_l^2}{2}\chi_{2L}^2, \quad (67)$$

where χ_{2L}^2 is a chi-square random variable with $2L$ degrees of freedom. For a given $T > 0$, P_{fa} is then computed by:

$$\begin{aligned}P_{fa} &= \left\{\sum_{l=0}^{L-1} z_{l,k} > T\right\} = \int_T^{+\infty} p(x)dx\\ &= \int_T^{+\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\phi(t)e^{-2\pi jtx}dt\right]dx\\ &= \frac{1}{2\pi}\int_T^{+\infty}\left[\int_{-\infty}^{+\infty}\frac{e^{-2\pi jtx}}{(1-jt\sigma^2\sum_{l=0}^{MK-1} w_l^2)^L}dt\right]dx\end{aligned}$$

$$= \sum_{s=0}^{L-1} \frac{\left(\frac{T}{\sigma^2 \sum_{l=0}^{MK-1} w_l^2}\right)^s}{s!} e^{-\frac{T}{\sigma^2 \sum_{l=0}^{MK-1} w_l^2}}. \quad (68)$$

This completes the proof of **Theorem 3**.

Proof of Theorem 4. For $x > 0$, define the function $f_n(x)$ by:

$$f_n(x) = \Pr\left\{\sum_{l=1}^{n} \frac{\sigma^2 \lambda_l}{2} \chi_2^2(l) > x\right\}, \quad (69)$$

where n is an integer in the range $1 \leq n \leq LN$. Since chi-square random variables are non-negative (i.e., their probability density functions are supported on the positive real axis), we have:

$$\begin{aligned} f_{n+1}(x) &= \Pr\left\{\sum_{l=1}^{n+1} \frac{\sigma^2 \lambda_l}{2} \chi_2^2(l) > x\right\} \\ &\geq \Pr\left\{\sum_{l=1}^{n} \frac{\sigma^2 \lambda_l}{2} \chi_2^2(l) > x\right\} = f_n(x). \quad (70) \end{aligned}$$

Also we have $f_1(x) = e^{-\frac{x}{\sigma^2 \lambda_1}}$ and

$$f_n(x) = \sum_{m=1}^{n} \frac{\lambda_m^{n-1}}{\prod_{1 \leq l \leq n, l \neq m} (\lambda_m - \lambda_l)} e^{-\frac{x}{\sigma^2 \lambda_m}}, \quad 2 \leq n \leq LN. \quad (71)$$

From the definition (15), we see that $f_n(T_n(P_{fa})) = P_{fa}$, $1 \leq n \leq LN$. Since $f_n(x) \leq f_{n+1}(x)$, $1 \leq n \leq LN-1$, and since $f_n(x)$ are decreasing functions of x, we see immediately that $T_n(P_{fa}) \leq T_{n+1}(P_{fa})$, $1 \leq n \leq LN-1$. Also $T_1(P_{fa}) = -\sigma^2 \lambda_1 \ln(P_{fa})$ and $T_{LN}(P_{fa}) = T$. This proves the chain of inequalities in (16).

We next prove the inequality (18). For each positive integer n, define $g_n(x)$ as follows:

$$g_n(x) = e^{-x}\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right). \quad (72)$$

It can be verified that

$$g_n'(x) = -\frac{x^n e^{-x}}{n!}, \quad (73)$$

and

$$g_n''(x) = \frac{x^{n-1}(x-n)e^{-x}}{n!}. \quad (74)$$

Clearly g_n is a decreasing function on the interval $(0, \infty)$, since $g_n'(x) < 0$ for $x > 0$. g_n is concave on $(0, n)$ and convex on (n, ∞) since $g_n''(x) < 0$ on $(0, n)$ and $g_n''(x) > 0$ on (n, ∞). It can be proved that $g_n(n) > 1/2$ for all $n \geq 1$ and $g_n(n)$ monotonically decreases to $1/2$ as n approaches infinity. Let the solution in x to the equation

$$g_n(x) = e^{-x} \sum_{l=0}^{n} \frac{x^l}{l!} = p, \quad 0 < p < 1, \quad (75)$$

be denoted by μ. We now derive an upper bound for μ. Define $\psi_n(x)$ as the natural logarithm of $g_n(x)$, that is,

$$\psi_n(x) = \ln(g_n(x)) = -x + \ln \sum_{l=0}^{n} \frac{x^l}{l!}. \quad (76)$$

The function $\psi_n(x)$ can be shown to be concave on the positive real line $(0, \infty)$. In fact,

$$\psi_n'(x) = -\frac{\frac{x^n}{n!}}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}, \quad (77)$$

and

$$\psi_n''(x) = -\frac{x^{n-1}\left[n + \sum_{k=1}^{n-1}\left(\frac{n}{k!} - \frac{1}{(k-1)!}\right)x^k\right]}{n!\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right)^2}. \quad (78)$$

The above expression shows that $\psi_n''(x) < 0$ for all $x \in (0, \infty)$. Hence $\psi_n(x)$ is concave on the positive real axis $(0, \infty)$ and is always below its tangent lines. Let $x_0 = n$. The equation of the tangent line of the curve of $\psi_n(x)$ at the point $(x_0, \psi_n(x_0))$ is given by:

$$y = (x - x_0)\psi_n'(x_0) + \psi_n(x_0). \quad (79)$$

As this tangent line is above the curve of $\psi_n(x)$, it follows that

$$(x - x_0)\psi_n'(x_0) + \psi_n(x_0) \geq \psi_n(x), \quad x \in (0, \infty). \quad (80)$$

Let μ_0 be the solution in x to the equation

$$\ln p = (x - x_0)\psi_n'(x_0) + \psi_n(x_0). \quad (81)$$

This linear equation in x is easily solved and μ_0 is obtained as:

$$\begin{aligned} \mu_0 &= x_0 + \frac{\ln p - \psi_n(x_0)}{\psi_n'(x_0)} \\ &= n + \frac{\ln p - \psi_n(n)}{\psi_n'(n)}. \quad (82) \end{aligned}$$

We claim that μ_0 is an upper bound for the solution μ to the equation (75). In fact, from (80) and (81), we see that $\ln p \geq \psi_n(\mu_0) = \ln(g_n(\mu_0))$. Hence $p \geq g_n(\mu_0)$. Since $g_n(\mu) = p$, we have $g_n(\mu) \geq g_n(\mu_0)$. Hence $\mu \leq \mu_0$, since $g_n(x)$ is a decreasing function of x. Thus μ_0 is an upper bound for μ.

We can simplify the expression (82). Assume $0 < p \leq 2/e$. Since $g_n(n)$ is a monotone decreasing sequence, $g_n(n) \leq g_1(1) = 2/e$ and therefore $\psi_n(n) \leq \ln(2/e)$. It follows that

$$\begin{aligned} &\mu_0 \quad (83) \\ &= n + \frac{\ln p - \psi_n(n)}{\psi_n'(n)} \leq n + \frac{\ln p - \ln(2/e)}{\psi_n'(n)} \\ &\leq n + \left[\frac{1 + n + \frac{n^2}{2!} + \frac{n^3}{3!} + \cdots + \frac{n^n}{n!}}{\frac{n^n}{n!}}\right] \ln\left(\frac{2}{pe}\right). \end{aligned}$$

This upper bound can be further simplified. Using the following result for large n,

$$\begin{aligned} &g_n(n) = \quad (84) \\ &e^{-n}\left(1 + n + \frac{n^2}{2!} + \frac{n^3}{3!} + \cdots + \frac{n^n}{n!}\right) \cong 1/2, \end{aligned}$$

the expression

$$\frac{1+n+\frac{n^2}{2!}+\frac{n^3}{3!}+\cdots+\frac{n^n}{n!}}{\frac{n^n}{n!}} \tag{85}$$

can be shown to be bounded from above by $\frac{\sqrt{2\pi n}}{2}$. In fact, using Stirling's formula

$$n! \cong \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \tag{86}$$

we see that

$$\begin{aligned}
&\left(\frac{1+n+\frac{n^2}{2!}+\frac{n^3}{3!}+\cdots+\frac{n^n}{n!}}{\frac{n^n}{n!}}\right) \\
&\approx \frac{\left(\frac{\exp(n)}{2}\right)}{\frac{n^n}{n!}} = \frac{n!\exp(n)}{2n^n} \\
&\approx \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\exp(n)}{2n^n} = \frac{\sqrt{2\pi n}}{2}.
\end{aligned} \tag{87}$$

Substituting (87) into (83), we obtain the following upper bound for μ:

$$\mu \le n + \frac{\sqrt{2\pi n}}{2}\ln\left(\frac{2}{pe}\right),\ 0 < p \le 2/e. \tag{88}$$

The inequality (18) then follows directly from (88).

Finally we prove (19). Since

$$\begin{aligned}
P_{fa} &= \Pr\left\{\sum_{l=0}^{L-1} z_{l,k} > T\right\} \\
&= \Pr\left\{\sum_{l=1}^{LN} \frac{\sigma^2\lambda_l}{2}\chi_2^2(l) > T\right\} \\
&\le \Pr\left\{\sum_{l=1}^{LN} \frac{\sigma^2\lambda_1}{2}\chi_2^2(l) > T\right\} \\
&= \Pr\left\{\frac{\sigma^2\lambda_1}{2}\chi_{2LN}^2 > T\right\} \\
&= \sum_{s=0}^{LN-1}\frac{\left(\frac{T}{\sigma^2\lambda_1}\right)^s}{s!}\ e^{-\frac{T}{\sigma^2\lambda_1}},
\end{aligned} \tag{89}$$

where χ_{2LN}^2 is a chi-square random variable of $2LN$ degrees of freedom, we have

$$P_{fa} \le \sum_{s=0}^{LN-1}\frac{\left(\frac{T}{\sigma^2\lambda_1}\right)^s}{s!}\ e^{-\frac{T}{\sigma^2\lambda_1}}. \tag{90}$$

This implies

$$\frac{T}{\sigma^2\lambda_1} \le B_{LN}(P_{fa}), \tag{91}$$

which completes the proof of (19).

Proof of Theorem 5. From (64) we see that

$$\begin{aligned}
P_{fa} &= \Pr\left\{\sum_{l=0}^{L-1} z_{l,k} > T\right\} \\
&= \Pr\left\{\sum_{m=1}^{N} \frac{\sigma^2\mu_m}{2}\chi_{2L}^2(m) > T\right\} \\
&\ge \Pr\left\{\frac{\sigma^2\mu_1}{2}\chi_{2L}^2(1) > T\right\} \\
&= \sum_{s=0}^{L-1}\frac{\left(\frac{T}{\sigma^2\mu_1}\right)^s}{s!}e^{-\frac{T}{\sigma^2\mu_1}}.
\end{aligned} \tag{92}$$

This implies that

$$\frac{T}{\sigma^2\mu_1} \ge B_L(P_{fa}) \quad \text{or} \quad \sigma^2\mu_1 B_L(P_{fa}) \le T. \tag{93}$$

Similarly from (64) we also see that

$$\begin{aligned}
P_{fa} &= \Pr\left\{\sum_{l=0}^{L-1} z_{l,k} > T\right\} \\
&= \Pr\left\{\sum_{m=1}^{N} \frac{\sigma^2\mu_m}{2}\chi_{2L}^2(m) > T\right\} \\
&\le \Pr\left\{\sum_{m=1}^{N} \frac{\sigma^2\mu_1}{2}\chi_{2L}^2(m) > T\right\} \\
&= \Pr\left\{\frac{\sigma^2\mu_1}{2}\chi_{2LN}^2 > T\right\} \\
&= \sum_{s=0}^{LN-1}\frac{\left(\frac{T}{\sigma^2\mu_1}\right)^s}{s!}e^{-\frac{T}{\sigma^2\mu_1}},
\end{aligned} \tag{94}$$

where χ_{2LN}^2 is a chi-square random variable with $2LN$ degrees of freedom. This implies that

$$\frac{T}{\sigma^2\mu_1} \le B_{LN}(P_{fa}) \quad \text{or} \quad T \le \sigma^2\mu_1 B_{LN}(P_{fa}). \tag{95}$$

Combining (93) and (95) yields (20).

It remains to prove (23). Rearrange the power levels $z_{l,k}$, $0 \le l \le L-1$, in increasing order as follows:

$$y_1 \le y_2 \le y_3 \le \cdots \le y_L. \tag{96}$$

Then we have

$$Ly_1 \le \sum_{l=0}^{L-1} z_{l,k} \le Ly_L, \tag{97}$$

and it follows that

$$\begin{aligned}
P_{fa} &= \Pr\left\{\sum_{l=0}^{L-1} z_{l,k} > T\right\} \qquad (98) \\
&\ge \Pr\{Ly_1 > T\} = \Pr\{y_1 > T/L\} \\
&= \Pr\{z_{l,k} > T/L,\ 1 \le l \le L-1\} \\
&= \prod_{l=0}^{L-1}\Pr\{z_{l,k} > T/L\} \\
&= \left[\sum_{m=1}^{N}\frac{\mu_m^{N-1}}{\prod_{1\le l\le N, l\ne m}(\mu_m-\mu_l)}e^{-\frac{\frac{T}{L}}{\sigma^2\mu_m}}\right]^L.
\end{aligned}$$

This implies

$$\sum_{m=1}^{N} \frac{\mu_m^{N-1}}{\prod_{1\le l\le N, l\ne m}(\mu_m-\mu_l)} e^{-\frac{\frac{T}{L}}{\sigma^2\mu_m}} \le P_{fa}^{1/L} \tag{99}$$

which in turn yields

$$\frac{T}{L} \ge T_1, \quad \text{or} \quad LT_1 \le T. \tag{100}$$

Similarly, we have

$$\begin{aligned} P_{fa} &= \Pr\left\{\sum_{l=0}^{L-1} z_{l,k} > T\right\} \qquad (101) \\ &\le \Pr\{Ly_L > T\} = \Pr\{y_L > T/L\} \\ &= 1-\Pr\{y_L \le T/L\} \\ &= 1-\Pr\{z_{l,k} \le T/L,\ 1\le l\le L-1\} \\ &= 1-\prod_{l=0}^{L-1}\Pr\{z_{l,k}\le T/L\} \\ &= 1-\prod_{l=0}^{L-1}\left[1-\Pr\{z_{l,k} > T/L\}\right] \\ &= 1-\left[1-\sum_{m=1}^{N}\frac{\mu_m^{N-1}}{\prod_{1\le l\le N, l\ne m}(\mu_m-\mu_l)} e^{-\frac{\frac{T}{L}}{\sigma^2\mu_m}}\right]^L. \end{aligned}$$

This implies

$$\begin{aligned} &\sum_{m=1}^{N} \frac{\mu_m^{N-1}}{\prod_{1\le l\le N, l\ne m}(\mu_m-\mu_l)} e^{-\frac{\frac{T}{L}}{\sigma^2\mu_m}} \\ &\ge 1-(1-P_{fa})^{1/L}, \end{aligned} \tag{102}$$

which in turn yields

$$\frac{T}{L} \le T_2, \quad \text{or} \quad T \le LT_2. \tag{103}$$

Combining (100) and (103) yields the inequalities of (23). The inequality (24) is obtained by using (102) and applying the argument in the proof of (19) in **Theorem 4**. The proof of **Theorem 5** is thus completed.

To prove **Theorem 6**, we need to prove two lemmas.

Lemma 1. Let $n \ge 2$. Assume $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$ is an increasing sequence of positive real numbers and define the sequence β_l by setting:

$$\beta_l = \frac{\alpha_l^{n-1}}{\prod\limits_{1\le m\le n, m\ne l}(\alpha_l-\alpha_m)}, \quad 1\le l\le n. \tag{104}$$

Then

$$\left\{\begin{array}{lcl} \sum_{l=1}^{n}\beta_l &=& 1, \\ \sum_{l=1}^{n}\frac{\beta_l}{\alpha_l} &=& 0, \\ \cdots & \cdots & \cdots \\ \sum_{l=1}^{n}\frac{\beta_l}{\alpha_l^{n-1}} &=& 0. \end{array}\right. \tag{105}$$

Proof of Lemma 1. Let $1 \le m \le n-1$ be an integer and define the polynomial P_m by $P_m(\alpha) = \alpha^m$. According to Lagrange's Interpolation Theorem, the following identity holds for all $\alpha \in (-\infty, +\infty)$:

$$\sum_{i=1}^{n} P_m(\alpha_i)\frac{\prod_{k\ne i}(\alpha_k-\alpha)}{\prod_{k\ne i}(\alpha_k-\alpha_i)} = P_m(\alpha), \tag{106}$$

or

$$\sum_{i=1}^{n} \alpha_i^m\frac{\prod_{k\ne i}(\alpha_k-\alpha)}{\prod_{k\ne i}(\alpha_k-\alpha_i)} = \alpha^m. \tag{107}$$

This identity can be rewritten as

$$\sum_{i=1}^{n}(-1)^{n-1}\frac{\beta_i}{\alpha_i^{n-1-m}}\prod_{k\ne i}(\alpha_k-\alpha) = \alpha^m. \tag{108}$$

If $m = n-1$, (108) can be rewritten as

$$\sum_{i=1}^{n}(-1)^{n-1}\beta_i\frac{\prod_{k\ne i}(\alpha_k-\alpha)}{\alpha^{n-1}} = 1. \tag{109}$$

Letting α go to infinity yields the identity

$$\sum_{i=1}^{n}(-1)^{n-1}\beta_i\lim_{\alpha\to+\infty}\frac{\prod_{k\ne i}(\alpha_k-\alpha)}{\alpha^{n-1}} = 1. \tag{110}$$

But

$$\lim_{\alpha\to+\infty}\frac{\prod_{k\ne i}(\alpha_k-\alpha)}{\alpha^{n-1}} = (-1)^{n-1}. \tag{111}$$

Substituting this result into (110) yields:

$$\sum_{i=1}^{n}\beta_i = 1. \tag{112}$$

This proves the first identity in (105). To prove the remaining identities in (105), assume $1 \le m \le n-1$ and let $\alpha = 0$ in (108). We then obtain the identity

$$\sum_{i=1}^{n}\frac{\beta_i}{\alpha_i^{n-1-m}}\prod_{k\ne i}\alpha_k = 0. \tag{113}$$

Dividing both sides of (113) by $\prod_{k=1}^{n}\alpha_k$ yields

$$\sum_{i=1}^{n}\frac{\beta_i}{\alpha_i^{n-m}} = 0. \tag{114}$$

This proves the rest of the identities in (105).

Lemma 2. Let $n \ge 2$, $0 < p < 1$ and let $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n$ be an increasing sequence of positive real numbers. Define

$$\left\{\begin{array}{l} C(p,\alpha_1,\alpha_2,\cdots,\alpha_n) \\ = \sum_{i=1}^{n-1}\frac{\alpha_n}{\alpha_i}\left[\frac{\alpha_n}{\alpha_i}-1\right]p^{\alpha_n/\alpha_i}\beta_i, \\ D(p,\alpha_1,\alpha_2,\cdots,\alpha_n) \\ = \sum_{i=1}^{n}\frac{\alpha_n}{\alpha_i}p^{\alpha_n/\alpha_i}\beta_i, \end{array}\right. \tag{115}$$

where β_i are defined by (104). Then

$$\left\{\begin{array}{l} C(p,\alpha_1,\alpha_2,\cdots,\alpha_n) < 0, \\ D(p,\alpha_1,\alpha_2,\cdots,\alpha_n) > 0. \end{array}\right. \tag{116}$$

Proof of Lemma 2. We prove this lemma by mathematical induction. First consider the case $n = 2$. We have

$$\begin{aligned} &C(p,\alpha_1,\alpha_2) = \qquad (117) \\ &\left[\frac{\alpha_2}{\alpha_1}\right]\left[\frac{\alpha_2}{\alpha_1}-1\right]p^{\alpha_2/\alpha_1}\beta_1 \\ &= \left[\frac{\alpha_2}{\alpha_1}\right]\left[\frac{\alpha_2}{\alpha_1}-1\right]p^{\alpha_2/\alpha_1}\left[\frac{\alpha_1}{\alpha_1-\alpha_2}\right] < 0, \end{aligned}$$

since $\alpha_2 > \alpha_1$ and $p \in (0,1)$. On the other hand,

$$\begin{aligned} & D(p,\alpha_1,\alpha_2) \\ &= \left[\frac{\alpha_2}{\alpha_1}\right]\left[p^{\alpha_2/\alpha_1}\right]\beta_1 + \left[\frac{\alpha_2}{\alpha_2}\right]\left[p^{\alpha_2/\alpha_2}\right]\beta_2 \\ &= \left[\frac{\alpha_2}{\alpha_1}\right]\left[\frac{\alpha_1}{\alpha_1-\alpha_2}\right]p^{\alpha_2/\alpha_1} - \left[\frac{\alpha_2}{\alpha_1-\alpha_2}\right]p \\ &= \left[\frac{\alpha_2}{\alpha_1-\alpha_2}\right]\left[p^{\alpha_2/\alpha_1} - p\right] > 0, \end{aligned} \tag{118}$$

again using the fact that $\alpha_2 > \alpha_1$ and $p \in (0,1)$. This proves **Lemma 2** for the case $n = 2$. Next assume **Lemma 2** holds for positive integer $n-1$. We show that **Lemma 2** must also hold for the integer n. To do this, we reformulate the expression $C(p,\alpha_1,\cdots,\alpha_n)$ as follows:

$$\begin{aligned} & C(p,\alpha_1,\alpha_2,\cdots,\alpha_n) \\ &= \sum_{i=1}^{n-1}\left[\frac{\alpha_n}{\alpha_i}\right]\left[\frac{\alpha_n}{\alpha_i}-1\right]p^{\alpha_n/\alpha_i}\beta_i \\ &= \sum_{i=1}^{n-1}\left[\frac{\alpha_n}{\alpha_i}\right]\left[\frac{\alpha_n-\alpha_i}{\alpha_i}\right]\frac{\alpha_i^{n-1}}{\prod_{k\neq i}(\alpha_i-\alpha_k)}p^{\alpha_n/\alpha_i} \\ &= -\alpha_n\sum_{i=1}^{n-1}\frac{1}{\alpha_i}\frac{\alpha_i^{n-2}}{\prod_{1\le k\le n-1,k\neq i}(\alpha_i-\alpha_k)}p^{\alpha_n/\alpha_i} \\ &= -\frac{\alpha_n}{\alpha_{n-1}}\sum_{i=1}^{n-1}\frac{\alpha_{n-1}}{\alpha_i}\left(p^{\alpha_n/\alpha_{n-1}}\right)^{\alpha_{n-1}/\alpha_i}\beta_i^*, \end{aligned} \tag{119}$$

where

$$\beta_i^* = \frac{\alpha_i^{n-2}}{\prod_{1\le k\le n-1,k\neq i}(\alpha_i-\alpha_k)}. \tag{120}$$

Hence

$$\begin{aligned} & C(p,\alpha_1,\alpha_2,\cdots,\alpha_n) = \\ & -\frac{\alpha_n}{\alpha_{n-1}}D(p^{\frac{\alpha_n}{\alpha_{n-1}}},\alpha_1,\alpha_2,\cdots,\alpha_{n-1}). \end{aligned} \tag{121}$$

Since $D(p,\alpha_1,\alpha_2,\cdots,\alpha_{n-1}) > 0$, we have

$$\begin{aligned} & C(p,\alpha_1,\alpha_2,\cdots,\alpha_n) = \\ & -\frac{\alpha_n}{\alpha_{n-1}}D(p^{\frac{\alpha_n}{\alpha_{n-1}}},\alpha_1,\alpha_2,\cdots,\alpha_{n-1}) < 0. \end{aligned} \tag{122}$$

To prove $D(p,\alpha_1,\alpha_2,\cdots,\alpha_n) > 0$, let

$$E(p,\alpha_1,\alpha_2,\cdots,\alpha_n) = \sum_{i=1}^{n} p^{\alpha_n/\alpha_i}\beta_i. \tag{123}$$

It can be verified that

$$\left\{\begin{array}{l} \frac{\partial E}{\partial p}(p,\alpha_1,\alpha_2,\cdots,\alpha_n) \\ = p^{-1}D(p,\alpha_1,\alpha_2,\cdots,\alpha_n), \\ \frac{\partial^2 E}{\partial^2 p}(p,\alpha_1,\alpha_2,\cdots,\alpha_n) \\ = p^{-2}C(p,\alpha_1,\alpha_2,\cdots,\alpha_n). \end{array}\right. \tag{124}$$

Since $C(p,\alpha_1,\alpha_2,\cdots,\alpha_n) < 0$, it follows that

$$\begin{aligned} & \frac{\partial^2 E}{\partial^2 p}(p,\alpha_1,\alpha_2,\cdots,\alpha_n) \\ & = p^{-2}C(p,\alpha_1,\alpha_2,\cdots,\alpha_n) < 0. \end{aligned} \tag{125}$$

Hence, $\frac{\partial E}{\partial p}(p,\alpha_1,\alpha_2,\cdots,\alpha_n)$, viewed as a function of p, is strictly decreasing on the interval $[0,1]$. Using the identity, $\sum_{i=1}^{n}\frac{\beta_i}{\alpha_i} = 0$, proved in Lemma 1, we see that

$$\begin{aligned} & \frac{\partial E}{\partial p}(p,\alpha_1,\alpha_2,\cdots,\alpha_n)|_{p=1} \\ &= \left(p^{-1}D(p,\alpha_1,\alpha_2,\cdots,\alpha_n)\right)_{p=1} \\ &= D(1,\alpha_1,\alpha_2,\cdots,\alpha_n) \\ &= \alpha_n\sum_{i=1}^{n}\frac{\beta_i}{\alpha_i} = 0. \end{aligned} \tag{126}$$

It follows that for all $0 < p < 1$,

$$\begin{aligned} & \frac{\partial E}{\partial p}(p,\alpha_1,\alpha_2,\cdots,\alpha_n) \\ & > \left(\frac{\partial E}{\partial p}(p,\alpha_1,\alpha_2,\cdots,\alpha_n)\right)|_{p=1} = 0, \end{aligned} \tag{127}$$

and consequently

$$\begin{aligned} & D(p,\alpha_1,\alpha_2,\cdots,\alpha_n) \\ & = p\left(\frac{\partial E}{\partial p}(p,\alpha_1,\alpha_2,\cdots,\alpha_n)\right) > 0. \end{aligned} \tag{128}$$

By the principle of mathematical induction, **Lemma 2** must hold for all integers $n \ge 2$.

Proof of Theorem 6. Applying **Lemma 2** to the increasing sequence $\lambda_{LN} < \lambda_{LN-1} < \cdots < \lambda_2 < \lambda_1$, we see immediately that $g'(p) > 0$ and $g''(p)$ for $0 < p < 1$.

Proof of Theorem 7. Since (c.f. (27))

$$\sum_{l=0}^{L-1} z_{l,k} \sim \sum_{m=1}^{N}\frac{\sigma^2\mu_m}{2}\chi^2_{2L}(m),$$

we see from Eq. (4.1) of [15] that

$$\begin{aligned} P_{fa} &= \Pr\left\{\sum_{l=0}^{L-1} z_{l,k} > T\right\} \\ &= \Pr\left\{\sum_{m=1}^{N}\mu_m\chi^2_{2L}(m) > 2T/\sigma^2\right\} \\ &\approx \Pr\left\{\chi^2_{h'} > y\right\}, \end{aligned} \tag{129}$$

where $h' = c_2^3/c_3^2$, $y = \left(2T/\sigma^2 - c_1\right)\left(h'/c_2\right)^{1/2} + h'$ and

$$\left\{\begin{array}{lcl} c_1 &=& 2L\sum_{m=1}^{N}\mu_m = 2LN, \\ c_2 &=& 2L\sum_{m=1}^{N}\mu_m^2, \\ c_3 &=& 2L\sum_{m=1}^{N}\mu_m^3. \end{array}\right. \tag{130}$$

Here we used the fact that $\sum_{m=1}^{N}\mu_m = N$, which follows from the assumption that the window is normalized. We have

$$h' = \frac{\left(2L\sum_{m=1}^{N}\mu_m^2\right)^3}{\left(2L\sum_{m=1}^{N}\mu_m^3\right)^2} = 2L\frac{\left(\sum_{m=1}^{N}\mu_m^2\right)^3}{\left(\sum_{m=1}^{N}\mu_m^3\right)^2}, \tag{131}$$

and

$$\begin{aligned}&(h'/c_2)^{1/2}\\&=\left[\frac{2L\frac{\left(\sum_{m=1}^{N}\mu_m^2\right)^3}{\left(\sum_{m=1}^{N}\mu_m^3\right)^2}}{2L\sum_{m=1}^{N}\mu_m^2}\right]^{1/2}=\frac{\sum_{m=1}^{N}\mu_m^2}{\sum_{m=1}^{N}\mu_m^3}. \qquad (132)\end{aligned}$$

It follows that

$$\begin{aligned}y&=\left(2T/\sigma^2-c_1\right)(h'/c_2)^{1/2}+h' \qquad (133)\\&=\left(2T/\sigma^2-2LN\right)\frac{\sum_{m=1}^{N}\mu_m^2}{\sum_{m=1}^{N}\mu_m^3}+2L\frac{\left(\sum_{m=1}^{N}\mu_m^2\right)^3}{\left(\sum_{m=1}^{N}\mu_m^3\right)^2}.\end{aligned}$$

Consequently

$$\begin{aligned}P_{fa}&\approx \Pr\left\{\chi_{h'}^2>y\right\}\approx \Pr\left\{\chi_{2h}^2>y\right\}\\&\approx \Pr\left\{\chi_{2h}^2/2>y/2\right\}\\&=e^{-z}\left(1+z+\frac{z^2}{2!}+\cdots+\frac{z^{h-1}}{(n-1)!}\right), \qquad (134)\end{aligned}$$

where $z=y/2$ and $h=\left\lfloor L\frac{\left(\sum_{m=1}^{N}\mu_m^2\right)^3}{\left(\sum_{m=1}^{N}\mu_m^3\right)^2}\right\rfloor$. Hence

$$z=y/2\approx B_h(P_{fa}),$$

or

$$\begin{aligned}&\left(T/\sigma^2-LN\right)\frac{\sum_{m=1}^{N}\mu_m^2}{\sum_{m=1}^{N}\mu_m^3}+L\frac{\left(\sum_{m=1}^{N}\mu_m^2\right)^3}{\left(\sum_{m=1}^{N}\mu_m^3\right)^2}\\&\qquad\approx B_h(P_{fa}). \qquad (135)\end{aligned}$$

The approximation (28) then follows immediately from (135).

The proof of **Theorem 8**, as it is almost identical to that of **Theorem 7**, is omitted here.

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