Adaptive-Gain Kinematic Filters of Orders 2-4

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***Abstract* - The kinematic filter is a common tool in control and signal processing applications dealing with position, velocity and other kinematical variables. Usually the filter gain is given a fixed value determined due to dynamic and measurement conditions. Most studies provide analytical solutions for optimal gains in particular scenarios. In practice, due to a lack of information (or under time-varying conditions) these recipes are mostly inapplicable and the kinematic filter requires appropriate adaptation tools instead. In its simplest form, the problem may be formulated as the gain adaptation under the tracking index uncertainty. We suggest a simple adaptive-gain kinematic filter based on minimization of the innovation variance which is known to give the optimal Kalman gain. The study deals with commonly used kinematic models of order 2-4. As shown, for any order of the kinematic filter its transfer function matches the moving-averaging (MA) model parameterized by the filter gain. In this view, the adaptive kinematic filter may be implemented in a variety of forms either based on the MA identification or by a direct gain adaptation. Optimal closed-form solutions may be incorporated into the adaptive filter as constraints. With the optimally constrained gain-vector components, the multiple-parameter adaptive filter is translated into a beneficial single-parameter version. The simulation study demonstrates behavior of suggested filters in a wide range of conditions.**

***Index Terms* - kinematic/tracking filter, adaptive gain, time-series identification**

I. INTRODUCTION

In many real-time computer-based applications such as target tracking, navigation and control the problem is usually interpreted in terms of kinematic variables, i.e., position and its derivatives. An important class of digital filters titled as kinematic, tracking, polynomial, position-velocity, etc remains the focus of interest over decades.

Kinematic trackers are commonly presented in a Kalman-like form when the time-varying Kalman scheme is reduced to a familiar α-β (or α-β-γ) steady-state filter with a somehow prefixed gain [1-6]. Many studies concentrate on the gain optimality w.r.t. the so-called tracking (maneuvering) index (TI) and some other conditions. In the literature, one can find numerous closed-form and numeric solutions providing optimal gains under various types of the process and observation noise, kinematic equations and other variations of the tracking problem.

In real-life situations, however, one meets uncertain or non-stationary conditions precluding direct use of such theoretical findings. Arrangement of a self-tuned adaptive kinematic filter is the issue of primary importance.

Historically this problem has been formulated in terms of adaptive Kalman filter (AKF) and treated by means of the process and measurement noise estimation. Jazwinsky [7] has considered the adaptive noise estimation problem as letting the noise input levels adapt to the residuals. Ohap [8] has developed a so-called adaptive minimum variance estimation method based on the direct variance estimation. Mehra [9-11], Gelb [12] and other authors also exploited on-line estimation of the innovation variance. Maybeck [13] gave detailed presentation of the adaptive Kalman filtering including the innovation based methods. This approach has been also explored, with such or other variations, in several recent works. In [14] (Mohamed et al.), estimation of the observation or process covariance is based on the empirical innovation. Similarly, in [15] the covariance-matching principle has been applied, first, to estimate the process covariance from the practical residuals and, secondly, to use it in the Kalman update equation. The method in [16] is to inspect the normalized autocorrelation of innovation. In [17], the innovation is matched with a stored replica of the innovation impulse response.

As an alternative to the innovation based AKF, the model uncertainty is frequently treated by a multiple-model method based on the multi-hypothesis approach [13]. Recently, much attention is drawn by the Monte-Carlo, particle filtering [18] and related methods. Another relevant field is the subspace identification [19, 20].

Together with all these powerful and yet sophisticated adaptation tools there is another candidate, namely, the parameter estimation method (PEM) whose efficiency depends on a properly parameterized model. Parameterization of the kinematic filter in terms of the kinematic gain is the core of the considered problem.

In general, the observer canonical form of the state-space model provides an autoregressive moving-average (ARMA) process [19] whose coefficients may be readily estimated. However, this method may be inefficient because of complex combinations of the gain and model parameters. Fortunately, the problem simplifies for the kinematic filter parameterized by the solo innovation gain.

In particular, the α-β, α-β-γ and higher order kinematic filters specify an autoregressive-integrated moving-average (ARIMA) series with parameters linearly tied to the kinematic gain. In this light, the kinematic gain may be tuned using principles of the above-mentioned PEM.

The objective of this paper is twofold. First, we suggest an approach which can be further extended by different methods. Some of them are mentioned in Section 7.

Secondly, the paper is oriented for practitioners interested in application of the α-β- or α-β-γ-like filters. We suggest two simple implementation schemes of the developed adaptive-gain kinematic filter. The first scheme identifies the (differenced) observation as a moving-average (MA) process and then translates its parameters into the desired kinematic gain. The state estimator is applied in cascade. In another implementation scheme the gain adjustment is coupled with the state estimator and relies on the innovation instead of the prediction error.

In the sequel, Section II recalls basics of the kinematic filter, Section III derives the transfer function and links between the innovation gain and MA model, Section IV presents the gain adaptation procedures, Section V analyzes the filter performance, Section VI describes the simulation study and Section VII concludes the work.

II. Filter Basics

Let us consider a discrete-time kinematic model

$$\mathbf{x}_{i+1} = \mathbf{F}\mathbf{x}_i + \mathbf{g}w_i$$
$$y_i = \mathbf{h}\mathbf{x}_i + v_i \quad (1)$$

where i – discrete time, $\mathbf{x}_i$ – $N\times1$ state-vector of kinematic variables, y_i – observation, $\mathbf{F}$ – $N\times N$ transition matrix, $\mathbf{h}$ – $1\times N$ measurement matrix, $\mathbf{g}$ – $N\times1$ control vector, w and v are mutually uncorrelated process and measurement noises, respectively, with variances $Q=\sigma_w^2$ and $R=\sigma_v^2$. The tracking filter follows the Kalman-like state-equation

$$\hat{\mathbf{x}}_{i+1} = \mathbf{F}\hat{\mathbf{x}}_i + \mathbf{k}_{i+1}e_{i+1} \quad (2a)$$

$$e_{i+1} = y_{i+1} - \mathbf{h}\mathbf{F}\hat{\mathbf{x}}_i \quad (2b)$$

where the superscript '^' marks the state estimator, e_i denotes innovation, $\mathbf{k}_i$ - gain-vector. The asymptotic gain $\mathbf{k}=\lim(\mathbf{k}_i)$ $(i\rightarrow\infty)$ is defined as [1]

$$\mathbf{k} = \mathbf{P}^{(-)}\mathbf{h}^T / S = \mathbf{P}^{(+)}\mathbf{h}^T / R \quad (3)$$

with the covariance update equations

$$\mathbf{P}^{(+)} = \mathbf{P}^{(-)} - \mathbf{k}\mathbf{h}\mathbf{P}^{(-)} \quad (4a)$$

$$\mathbf{P}^{(-)} = \mathbf{F}\mathbf{P}^{(+)}\mathbf{F}^T + \mathbf{g}Q\mathbf{g}^T \quad (4b)$$

where $\mathbf{P}^{(+)}=\{p^{(+)}_{i,j}\}$ and $\mathbf{P}^{(-)}=\{p^{(-)}_{i,j}\}$, $i,j=1,\ldots,N$ are the estimation and prediction covariances, respectively, and

$$S = E\left\langle \left(y_{i+1} - \mathbf{h}\mathbf{F}\hat{\mathbf{x}}_i\right)^2 \right\rangle = \mathbf{h}\mathbf{P}^{(-)}\mathbf{h}^T + R \quad (5)$$

is the innovation variance (E – expectation sign). The minimum variance of e_i, irrespective of the filter order, is

$$S_o = \min_{\mathbf{k}} S = \sigma_v^2 / (1-\alpha) \quad (6)$$

There are particular recipes for the kinematic gain $\mathbf{k}$.

A. α-β Filter

The so-called α-β filter associates with the model

$$\mathbf{x}_i = \begin{bmatrix} x_i \\ \dot{x}_i \end{bmatrix}, \mathbf{F} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}, \mathbf{g} = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}, \mathbf{h} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T, \mathbf{k} = \begin{bmatrix} \alpha \\ \beta/T \end{bmatrix} \quad (7)$$

where x_i and its first derivative are the position and velocity states, T - sampling period. The steady gain $\mathbf{k}$ is defined by a vector with normalized terms α and β. Given a so-called tracking index Λ (Kalata [3])

$$\Lambda = \sigma_w T^2 / \sigma_n \quad (8)$$

one can derive optimal α and β as [4]:

$$\alpha = -0.125\left(\Lambda^2 + 8\Lambda - (\Lambda+4)\sqrt{\Lambda^2+8\Lambda}\right)$$
$$\beta = 0.25\left(\Lambda^2 + 4\Lambda - \Lambda\sqrt{\Lambda^2+8\Lambda}\right) \quad (9)$$

The optimal α and β are tied by the useful link

$$\beta = 2(2-\alpha) - 4\sqrt{1-\alpha} \quad (10)$$

B. α-β-γ Filter

Another common filter, α-β-γ, relies on the model

$$\mathbf{x}_i = \begin{bmatrix} x_i \\ \dot{x}_i \\ \ddot{x}_i \end{bmatrix}, \mathbf{F} = \begin{pmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{g} = \begin{bmatrix} T^2/2 \\ T \\ 1 \end{bmatrix}, \mathbf{h} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T, \mathbf{k} = \begin{bmatrix} \alpha \\ \beta/T \\ \gamma/2T^2 \end{bmatrix} \quad (11)$$

where the 3-state vector $\mathbf{x}_i$ comprises, in addition, the acceleration state and the gain-vector $\mathbf{k}$ contains an auxiliary term $\gamma/2T^2$. The link (10) between optimal α and β holds while the optimal γ follows as [3]

$$\gamma = \beta^2 / 4\alpha \quad (12)$$

C. α-β-γ-λ Filter

A 4-state constant-jerk kinematic model is defined as [4]

$$\mathbf{x}_i = \begin{bmatrix} x_i \\ \dot{x}_i \\ \ddot{x}_i \\ \dddot{x}_i \end{bmatrix}, \mathbf{F} = \begin{pmatrix} 1 & T & T^2/2 & T^3/6 \\ 0 & 1 & T & T^2/2 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{g} = \begin{bmatrix} T^3/6 \\ T^2/2 \\ T \\ 1 \end{bmatrix}, \mathbf{h} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (13a)$$

Similar to lower orders the 4-state filter, let it be titled as α-β-γ-λ, may be used with a gain-vector of the form

$$\mathbf{k} = \begin{pmatrix} \alpha & \beta/T & \gamma/2T^2 & \lambda/6T^3 \end{pmatrix}^T \quad (13b)$$

Closed-form solutions in this case are unavailable while a numeric procedure may be found, e.g., in [4].

For convenience, the gain may be expanded as

$$\mathbf{k} = \mathbf{Z}\boldsymbol{\chi} \quad (14)$$

where $\boldsymbol{\chi}=[\alpha\ \beta\ \gamma\ \ldots]^T$ is introduced as a normalized gain and $\mathbf{Z}=\mathrm{diag}(1\ 1/T\ 1/2T^2\ \ldots)$ - a proper scaling diagonal matrix.

So, given Λ one may find optimal $\boldsymbol{\chi}$ (and $\mathbf{k}$) analytically or numerically. In practice, however, Λ is unknown or time-varying term. Accordingly, we consider an adaptive technique allowing on-line adjustment of the gain $\mathbf{k}$.

III. Transfer Function

In the first step we translate the state-space model into a properly parameterized transfer function (t.f.). One may tie the innovation and observation by the t.f. [19]

$$W = 1 + \mathbf{h}\left[q\mathbf{I} - \mathbf{F}\right]^{-1}\mathbf{F}\mathbf{k} \quad (15)$$

where q denotes the forward shift operator. Since the matrix $\mathbf{F}$ and vector $\mathbf{h}$ are constant the t.f. (15) is parameterized by the solo $\mathbf{k}$. Expanding (15) shows that the resulting t.f. has a common for all orders N form

$$W = D_N / \Delta^{(N)} \quad (16)$$

which describes the N-order ARIMA process driven by the innovation e_i. $\Delta^{(N)}$ in (16) denotes the N-order difference and $D_N=D_N(q)$ is a polynomial of q^{-1}, namely

$$D_2 = 1 + (\alpha + \beta - 2)q^{-1} + (1-\alpha)q^{-2} \quad (17a)$$

$$D_3 = 1 + (\alpha + \beta + \frac{\gamma}{4} - 3)q^{-1} + (3 - 2\alpha - \beta + \frac{\gamma}{4})q^{-2} + (\alpha - 1)q^{-3} \quad (17b)$$

$$D_4 = 1 + \left(\alpha + \beta + \frac{\gamma}{4} + \frac{\lambda}{36} - 4\right)q^{-1} + \left(\frac{\lambda}{9} - 3\alpha - 2\beta + 6\right)q^{-2} + \left(3\alpha + \beta - \frac{\gamma}{4} + \frac{\lambda}{36} - 4\right)q^{-3} + (1-\alpha)q^{-4} \quad (17c)$$

Accordingly, (16) yields the time-domain MA equation

$$\Delta^{(N)} y_i = D_N \varepsilon_i = \varepsilon_i + \sum_{j=1}^{N} b_j \varepsilon_{i-j} \quad (18)$$

where b_j, j=1,…, N denote the MA parameters and the prediction error ε_i replaces the innovation e_i. In view of (17) $\boldsymbol{\chi}$ and vector $\mathbf{b}=[b_1,\ldots, b_N]^T$ are tied by a linear link

$$\mathbf{b} = \mathbf{L}\boldsymbol{\chi} + \mathbf{b}_0 \quad (19)$$

where for orders N=2 to 4, respectively, **L** and $\mathbf{b}_0$ are:

$$for\ N=2: \quad \mathbf{L} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{b}_0 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (20a)$$

$$for\ N=3: \quad \mathbf{L} = \begin{pmatrix} 1 & 1 & 1/4 \\ -2 & -1 & 1/4 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{b}_0 = \begin{pmatrix} -3 \\ 3 \\ -1 \end{pmatrix} \quad (20b)$$

$$for\ N=4: \quad \mathbf{L} = \begin{pmatrix} 1 & 1 & 1/4 & 1/36 \\ -3 & -2 & 0 & 1/9 \\ 3 & 1 & -1/4 & 1/36 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{b}_0 = \begin{pmatrix} -4 \\ 6 \\ -4 \\ 1 \end{pmatrix} \quad (20c)$$

For all N inverse of **L** exists and $\boldsymbol{\chi}$ follows from (19) as

$$\boldsymbol{\chi} = \mathbf{L}^{-1}(\mathbf{b} - \mathbf{b}_0) \quad (21)$$

The vector $\boldsymbol{\chi}$ (and yet **k**) is uniquely tied to **b**. Two adaptation methods for **k** result from the above discussion.

VI. Adaptive Filter

A. *Method 1*

Summarizing the Method 1 (sketched in Fig. 1), one finds the N^{th} difference of y_i, estimates the vector **b** for the N^{th}-order MA model and maps **b** into the gain vector **k**. The state estimator utilizes the most recent value of **k**. The MA identification is thus viewed as a stand-alone task which can run at a rate different to that of the state estimator.

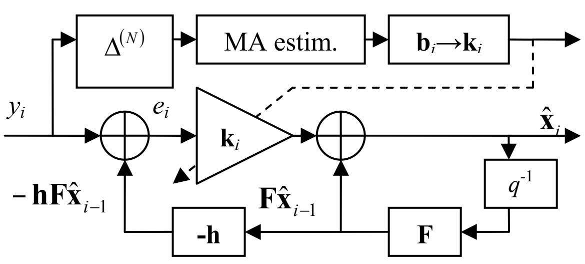

Fig. 1. Block diagram of the Method 1.

The MA identification is a familiar regression requiring use of the so-called regressors (sensitivity functions) [19]

$$\boldsymbol{\Psi} = \frac{\partial \varepsilon_i}{\partial \mathbf{b}} = -\left[q^{-N}\varepsilon_i^F\right], \quad \left(\frac{\partial \varepsilon_i}{\partial b_m} = -q^{-m}\varepsilon_i^F,\ m = 1,2,\ldots,N\right) \quad (22)$$

where, in view of (18), the filtered error [20] is defined as

$$\varepsilon_i^F = \varepsilon_i / D_N(q) \quad (23)$$

The Method 1 allows variants. The Matlab user may apply standard routines RARMAX or ARMAX for the MA identification. Thus the batch ARMAX routine can be used periodically after collecting groups of data (Table 1).

Table 1. Estimation of the kinematic gain.

Step Definition	MATLAB code
N-difference of y	yd=diff(*y*,N)
Batch MA estimator	MDL=armax(yd,[0,N])
Collect **b**	B(1:N)=MDL.c(2:N+1)
Find gain $\boldsymbol{\chi}$	kn=inv(L)*(b-b0)’ %Eq.(21)
Find gain **k**	k=Z*kn %Eq.(14)

Accordingly the time-varying $\mathbf{k}_i$ follows as a piecewise-constant term. Alternatively one may replace the second and third raws in Table 1 by the single line:

```
[b,yHat]=rarmax(yd,[0,N])
```

The RARMAX routine updates **b** at each instant so the whole procedure immitates on-line estimation of $\mathbf{k}_i$.

Note that at an initial stage only the MA estimation may be performed until the kinematic gain approaches its optimum. Neither the vector of kinematic variables nor the innovation is necessary for the MA estimator. Given appropriately tuned $\mathbf{k}_i$ one may start (or continue) the kinematic state estimator. Initialization of the position and other states is performed as usual [1]. The state errors do not affect $\mathbf{k}_i$.

The Method 1 distinguishes between the prediction error and innovation. The former associates with the MA predictor while the latter with the state-space predictor.

B. *Method 2*

With the Method 2 (Fig. 2), the MA identification step is omitted and χ_i is updated explicitly, using the innovation instead of the prediction error. Keeping in mind (19) one determines derivatives of the innovation w.r.t. $\boldsymbol{\chi}=\boldsymbol{\chi}_i$ as

$$\mathbf{H} = \frac{\partial e_i}{\partial \boldsymbol{\chi}} = \frac{\partial e_i}{\partial \mathbf{b}} \cdot \frac{\partial \mathbf{b}}{\partial \boldsymbol{\chi}} = \mathbf{L}\frac{\partial e_i}{\partial \mathbf{b}} \quad (24)$$

In view of (22), the regressor (24) obtains the form

$$\mathbf{H} = -\mathbf{L}q^{-N}\xi_i \quad (25)$$

where $\xi_i=e_i/D_N(q)$ is the filtered innovation obtained by analogy to the filtered prediction error (23).

Regressors for N=2-4 are specified in Table 2.

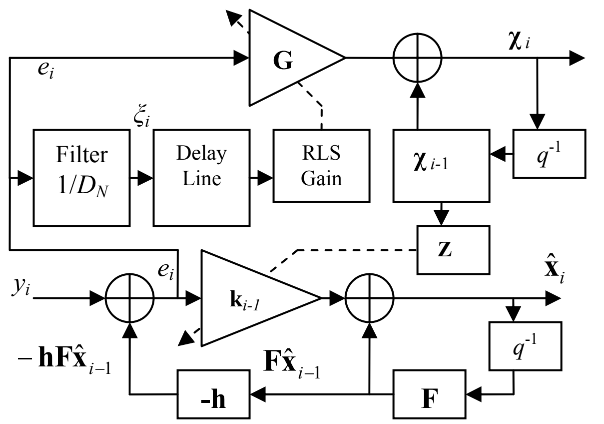

Fig. 2. Block diagram of the Method 2.

Table 2. Sensitivity functions w.r.t. terms of χ (N=2-4)

N / H	2	3	4
$\partial e_i/\partial\alpha$	$-q^{-1}\nabla\xi_i$	$-q^{-1}\nabla^{(2)}\xi_i$	$-q^{-1}\nabla^{(3)}\xi_i$
$\partial e_i/\partial\beta$	$-q^{-1}\xi_i$	$-q^{-1}\nabla\xi_i$	$-q^{-1}\nabla^{(2)}\xi_i$
$\partial e_i/\partial\gamma$		$-0.25\left(q^{-1}+q^{-2}\right)\xi_i$	$-0.25\left(q^{-1}-q^{-3}\right)\xi_i$
$\partial e_i/\partial\lambda$			$-\left(1/36\right)\left(q^{-1}-4q^{-2}+q^{-3}\right)\xi_i$

Table 3. Adaptive-Gain Kinematic Filter

Design Parameters:
Initial and Final Convergence Factors $\eta_0=\eta(0)$, $\eta_\infty=\eta(\infty)$
Model Order N, Forgetting Factor μ
Transition Matrix **F** *(due to Eq. (7), (11) or (13))*
Initial Matrix **P**(0), *Initial Normalized Gain* χ (0)
Initialization:
Convergence Factor: $\eta=\eta_0$
Inverse Covariance Matrix: **P**=**P**(0)
Kinematic State Vector: $\mathbf{x}_0=\mathbf{x}(0)$
Kinematic Gain: $\chi_0=\chi(0)$, $\mathbf{k}_0=\mathbf{Z}\chi_0$
Compute For Time Instant i=1, 2, …
------------------------*State Estimator* ---------------
$e_i=y_i-\mathbf{hFx}_{i-1}$ *Eq. (2): Innovation*
$\mathbf{x}_i=\mathbf{Fx}_{i-1}+\mathbf{k}_{i-1}e_i$ *Eq. (2): State Update*
-------------------------*RLS*---------------------------
$\Psi=(-\xi_{i-1},\ldots,-\xi_{i-N})$ *Eq. (22)*
$\mathbf{H}=\mathbf{L}\Psi$ *Eq. (25)*
$\mathbf{G}=[\mathbf{HPH}^T+\eta]^{-1}\mathbf{PH}^T$ *RLS gain*
$\mathbf{P}=\eta^{-1}[\mathbf{P}-\mathbf{GHP}]$ *Inverse Covariance*
$\chi_i=\chi_{i-1}+\mathbf{G}e_i$ *Parameter Update*
$\mathbf{k}_i=\mathbf{Z}\chi_i$ *Tracker Gain Eq.(14)*
$\mathbf{b}=\mathbf{L}\chi_i$ *Eq. (19)-Translate* χ *into* **b**
$\xi_i=e_i-b_1\xi_{i-1}-\ldots-b_N\xi_{i-N}$ *Eq. (36),* $\xi_j=0, j<1$
$\eta=\mu\gamma+(1-\mu)\eta_\infty$ *Convergence Factor Update*

Table 3 describes the adaptive-gain tracker built in accordance to the Method 2. The algorithm has two basic stages. The first stage is the state update based on a given gain $\mathbf{k}_{i-1}$. The second is a usual RLS which refreshes the desired gain $\mathbf{k}_i$ (not to be confused with the RLS gain **G**).

The second stage exploits the current innovation obtained in the first stage. The first stage receives $\mathbf{k}_{i-1}$ from the second stage fulfilled at the preceding instant. Method 2 utilizes the same innovation to update the state vector and to refresh the adaptive gain $\mathbf{k}_i$.

To determine the required sensitivity functions, one collects the delayed filtered errors ξ_i and translates them into desired terms using the predefined matrix **L**.

C. Constrained Adaptive Filter

It is noteworthy that the optimality constraints (10) and (12) can be properly aggregated with the adaptive filter. With the Method 1, however, these constraints should be used in the MA identification and this takes us out of the attractive standard MA estimator. In the Method 2 use of constraints is straightforward and its description follows.

First, we consider the α-β adaptive kinematic filter. The derivatives of e_i w.r.t. α and β (see Table 2) specify a typical two-parameter adaptative scheme. The number of adapted parameters, however, can be reduced using (10). Viewing β as a function of α provides

$$\frac{\partial e_i}{\partial\alpha}=-\left(1+\beta'\right)\xi_{i-1}+\xi_{i-2}=-q^{-1}\Delta\xi_i-\beta'\xi_{i-1} \quad (26)$$

where the derivative for β follows from (10) as

$$\beta'=-2+2\big/\sqrt{1-\alpha} \quad (27)$$

Applying the gradient (26) together with a proper expression (10) for β results in a single-parameter scheme.

To give the constrained adaptive filter even a simpler truncated (β'=0) form one can drop the extra term in (26).

Now we turn to the α-β-γ adaptive kinematic filter.

The derivatives of e_i w.r.t. α, β and γ present in Table 2 specify a 3-parameter adaptive tracker. Analogously, one may rearrange it into a single-parameter form. Viewing β and γ as functions of α gives

$$\frac{\partial e_i}{\partial\alpha}=-(1+\beta'+\frac{1}{4}\gamma')\xi_{i-1}-(\frac{1}{4}\gamma'-\beta'-2)\xi_{i-2}-\xi_{i-3} \quad (28)$$

where β' is held as in (27) while γ' follows from (12) as

$$\gamma'=\partial\gamma/\partial\alpha=\beta\left(2\alpha\beta'-\beta\right)/\alpha^2 \quad (29)$$

Letting $\beta'=\gamma'=0$ in (28) yields a truncated filter form.

V. Performance Analysis

The adaptive tracker inherits many attractive properties of its predecessors, the Kalman filter and MA estimator. The asymptotic error variance of the recursive MA is equivalent to that of the batch LS estimator and this simplifies analysis of the kinematic filter. The covariance of the gain errors should achieve the Cramer-Rao bound.

Importantly that due to the known unimodality property of the MA estimator, the MA-based adaptive tracker necessarily converges to a true optimal-gain solution. Some of these points will be next discussed in more detail.

A. Gain Optimality

The first point to be stressed is that the minimum-innovation-variance yields the optimal steady-state Kalman gain. As known [8] if the transition matrix **F** and observation matrix **h** are available, the steady gain **k** which whitens the innovation process (and minimizes the innovation variance) is the optimal steady Kalman gain.

The Kalman gain (3) obeys the optimality condition

$$\frac{\partial Tr\left[\mathbf{P}^{(+)}\right]}{\partial \mathbf{k}} = 0 \qquad (30)$$

where '*Tr*' is short for the matrix trace,

$$Tr\left[\mathbf{P}^{(+)}\right] = \sum_{1}^{N} p_{i,i}^{(+)} \qquad (31)$$

On the second hand, minimizing S [defined in (5)] gives

$$\frac{\partial S}{\partial \mathbf{k}} = \frac{\partial p_{1,1}^{(-)}}{\partial \mathbf{k}} = 0 \qquad (32)$$

As known, the Kalman gain is optimal for any linear combination of states, including one underlying (32). In particular, the equivalence of (30) and (32) for the *α-β* filter can be inspected explicitly.

The innovation variance may be expanded as

$$p_{1,1}^{(-)} = p_{1,1}^{(+)} + p_{2,2}^{(+)}T^2 + 2p_{1,2}^{(+)}T + QT^4/4 \qquad (33)$$

Assume that $\mathbf{P}^{(-)}$ is a steady-state covariance associated with the optimal *α-β* filter. Then $\mathbf{P}^{(+)}$ may be obtained as

$$\begin{aligned} p_{1,1}^{(+)} &= (1-\alpha)^2 p_{1,1}^{(-)} + \alpha^2 R \\ p_{2,2}^{(+)} &= (\beta/T)^2\left(p_{1,1}^{(-)} + R\right) - 2(\beta/T)p_{1,2}^{(-)} + p_{2,2}^{(-)} \\ p_{1,2}^{(+)} &= \left[p_{1,2}^{(-)} - (\beta/T)p_{1,1}^{(-)}\right](1-\alpha) + R\alpha(\beta/T) \end{aligned} \qquad (34)$$

Differentiating the latter w.r.t. α and β and collecting non-zero derivatives provides (up to a scale factor)

$$\begin{aligned} \frac{\partial p_{1,1}^{(+)}}{\partial \alpha} &= \frac{\partial p_{1,2}^{(+)}}{\partial \beta} \propto -p_{1,1}^{(-)}(1-\alpha) + R\alpha \\ \frac{\partial p_{2,2}^{(+)}}{\partial \beta} &= \frac{\partial p_{1,2}^{(+)}}{\partial \alpha} \propto -\left[p_{1,2}^{(-)} - (\beta/T)p_{1,1}^{(-)}\right] + (\beta/T)R \end{aligned} \qquad (35)$$

Equating the first equation (35) to zero gives the Kalman

$$\alpha = p_{1,1}^{(-)}/\left(p_{1,1}^{(-)} + R\right) \qquad (36)$$

while the second equation (35) gives, in turn, the Kalman

$$\beta/T = p_{2,1}^{(-)}/\left(p_{1,1}^{(-)} + R\right) \qquad (37)$$

Note, the third line (34) (cross covariance) combines the optimality conditions both for α and β.

So, the loss functions (31) and (33) are minimized by the same α and β. Similarly, one may illustrate equivalence between criteria (30) and (32) for higher-order filters.

B. Unimodality

There is always a question of interest whether the asymptotic innovation variance is a convex unimodal function of adjusted parameters.

Regarding the ARMA identification there is a unique minimum if the ARMA model order is correctly chosen [20]. In this case the ARMA estimator provides the true solution. Accordingly if the kinematic model order is correctly chosen the adaptive tracker due to its unique correspondence with the MA model finds the 'true' gain.

As an example, let us depict variances of the predictor and estimator for the suggested constrained *α-β* filter.

The state estimator error of this filter is extended as

$$\tilde{x}_i \triangleq \hat{x}_i - x_i = (1-\alpha)\left(\frac{T^2}{2}\right)\frac{1+q^{-1}}{D_2(q)}w_i + \frac{\alpha + (\beta-\alpha)q^{-1}}{D_2(q)}v_i \qquad (38)$$

The state error (38) is built from two parts induced by the process and measurement noises. Note that strength of the fist part is proportional to $\sigma_w T^2$, strength of the second – to σ_v, while their ratio gives exactly the index Λ determined by (8). As α varies so the strength of 'dynamic' and 'measurement' parts vary in opposite directions.

The prediction error (innovation) is also composed from two parts induced by the process and measurement noises,

$$e_i = W^{-1}(x_i + v_i) = (1-\alpha)\left(\frac{T^2}{2}\right)\frac{1+q^{-1}}{D_2(q)}w_i + \frac{\Delta^{(2)}}{D_2(q)}v_i \qquad (39)$$

The independent terms in (38)-(39) fit the 2nd-order ARMA model [23]

$$\zeta_i = \frac{b_b q^{-1} + b_2 q^{-2}}{1 + a_1 q^{-1} + a_2 q^{-2}}\omega_i \qquad (40)$$

driven by the unit-strength noise ω_i. The variance of ζ_i is

$$E\left\langle \varsigma^2 \right\rangle = \frac{B_0(1+a_2) - B_1 a_1 + B_2\left(a_1^2 - a_2 - a_2^2\right)}{(1-a_2)\left[(1+a_2)^2 - a_1^2\right]} \qquad (41)$$

$$B_0 = 1 + b_1^2 + b_2^2,\ B_1 = 2(b_1 + b_1 b_2),\ B_2 = 2b_2$$

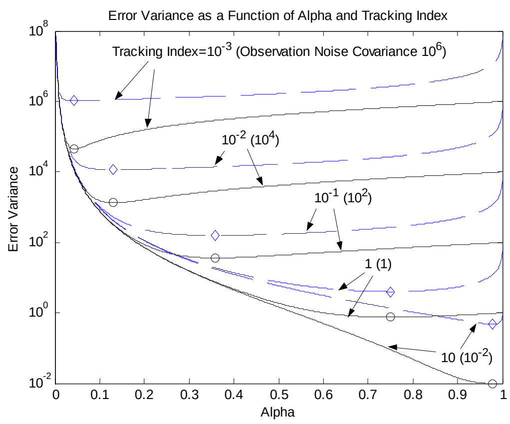

Fig. 3. Estimator error variance (solid curve) and innovation variance (dash) as functions of α and Λ. Diamonds and circles indicate minima of the innovation and estimation error variance respectively.

In the case of *α-β* filter a_l, l=1, 2; are zeros while b_j, j=1,2; are functions (19) of α and β. By substituting (10) β may be excluded. Fig. 3 depicts numerically computed variances of the estimator error and innovation as functions of α for different Λ. As expected, for any Λ both variances are unimodal and achieve their minima with the same α. Note, for higher Λ (smaller α) the innovation variance is more sensitive to α and has a more stressed minimum than the estimator. This means that a slight deviation from the optimal α caused by the gain misadjustment noise is not crucial for the kinematic state-estimator. As Λ decreases, α grows and its accuracy becomes more critical.

C. Error Covariance for Multiple-Parameter Filter

Accuracy of $\boldsymbol{\chi}$ is measured by the covariance $\mathbf{V}(\boldsymbol{\chi})$ which can be found from the covariance matrix for the MA parameters $\mathbf{V}(\mathbf{b})$. Specifically,

$$\mathbf{V}(\alpha,\beta) = n^{-1}\mathbf{L}^{-1}\mathbf{V}(\mathbf{b})(\mathbf{L}^{-1})^T \quad (42)$$

where n is an equivalent length of the data window. The 2nd-order MA estimator has the covariance matrix [22]

$$\mathbf{V}(b_1,b_2) = n^{-1}\begin{bmatrix} 1-b_2^2 & -b_1(1+b_2) \\ -b_1(1+b_2) & 1-b_2^2 \end{bmatrix} \quad (43)$$

Accordingly, applying (42), the α-β filter has the covariance matrix (in terms of b_1 and b_2)

$$\mathbf{V}(\alpha,\beta) = n^{-1}\begin{bmatrix} b_1(1+b_2) & b_1(1+b_2)-(1-b_2^2) \\ b_1(1+b_2)-(1-b_2^2) & 2\{1-b_2^2-b_1(1+b_2)\} \end{bmatrix} \quad (44)$$

where b_1 and b_2, as noted, are known functions of α and β.

D. Error Covariance for Single-Parameter Version

The optimal solution for $\boldsymbol{\chi}$ varies in accordance to Λ. The constraints (10) and (12) describe a subspace of solutions satisfying the optimality criterion and hence their use doesn't change the optimal solution. For the 2-state filter the space of optimal solutions is a curve in the α-β plane. The particular Λ specifies a point in this curve.

Though constraints do not change the optimal solution they do reduce the estimator variance. For the single-parameter adaptive α-β filter the variance for α follows as

$$\mathbf{V}(\alpha) = n^{-1}\left\{\begin{bmatrix}1 & \beta'\end{bmatrix}\mathbf{V}^{-1}(\alpha,\beta)\begin{bmatrix}1 \\ \beta'\end{bmatrix}\right\}^{-1}$$

$$= n^{-1}\frac{(b_2+1)(b_2-1+b_1)(b_2-1-b_1)}{2\beta'(b_2-1-b_1)+(\beta')^2(1-b_2)-2(b_2-1-b_1)} \quad (45)$$

Keeping in mind (20a) and (27) one can readily compute variances for both multiple- and single-parameter versions of the adaptive tracking filter. Comparison shows that Eq. (45) provides a lower than (44) variance of α.

VI. Simulation

The simulation study pursues several objectives. First, it inspects behavior of various adaptive filter types. Next, the study compares the adaptive gain to analytically and numerically found optimal solutions. The multi-parameter filter is checked versus its single-parameter variant.

A. Signal Model

The observation is viewed as the output of the system

$$y_i = \mathbf{h}(q\mathbf{I}-\mathbf{F})^{-1}\mathbf{g}w_i + v_i \quad (46)$$

excited by the input noise w_i and contaminated by the observation noise v_i. The latter model obtains particular forms in accordance to the specified kinematic matrix $\mathbf{F}$ and column vector $\mathbf{g}$. Note that $\mathbf{g}$ may specify different types of inputs. Thus Eq. (7) defines a so-called piecewise-constant random noise assuming that the input is constant between samples, while (11) and (13) provide a quantized instantaneous random noise [6].

For the 2nd-order model Eq. (46) is expanded as

$$y_i = \left[0.5T^2(1+q^{-1})/(1-q^{-1})^N\right]w_i + v_i \quad (47)$$

with N=2. The 3-state model (11) driven by the input random noise results in a similar to (47) form however with N=3. In turn, the 4-state model (13) defines y_i as

$$y_i = (T^2/6)\left[(1+4q^{-1}+q^{-2})/(1-q^{-1})^4\right]w_i + v_i \quad (48)$$

The noises w_i and v_i are generated as white Gaussian sequences. Let us recall that the optimal kinematic gain depends exclusively on Λ. Hence we may fix any of variances (let it be Q=1) and vary the second in accordance to the desired Λ. In all trials assumed T=1.

B. Simulation Results

In the first example we compare the Methods 1 and 2. Three variants of the α-β filter are involved. The first variant (as a reference) represents the Method 1 exploiting the batch ARMAX routine for the gain computation via the MA parameters. The Method 1 utilizing the recursive RARMAX routine for the gain computation via the MA parameters is the second variant. The Method 2 coupling direct adaptation of α and β with the state estimator is the third variant.

As seen from Fig. 4, both recursive estimators (second and third variants) rapidly approach the batch LS estimator (first variant) and yet the second variant converges faster. The reason is that with the Method 1 behind the second variant the gain (provided by the MA estimator and corresponding linear transformation of the MA parameters) is not affected by the kinematic state errors.

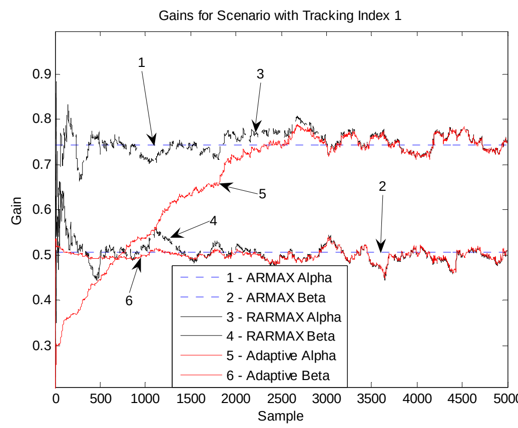

Fig. 4. Tracker gains: 1,2 – Method 1 (ARMAX), 3,4 – Method 1 (RARMAX), 5,6 – Method 2 (Mult.- Par.).

The second example inspects performance of the constrained variant of the kinematic filter. Fig. 5 presents behavior of the constrained and unconstrained variants of the α-β filter implemented in accordance with the Method 2 (the gain adaptation and state estimator are coupled). Both constrained and unconstrained recursive versions rapidly approach the batch form of the Method 1 (the same as in the previous example). However, the gain α converges faster and has a lower variance with the constrained single-parameter filter.

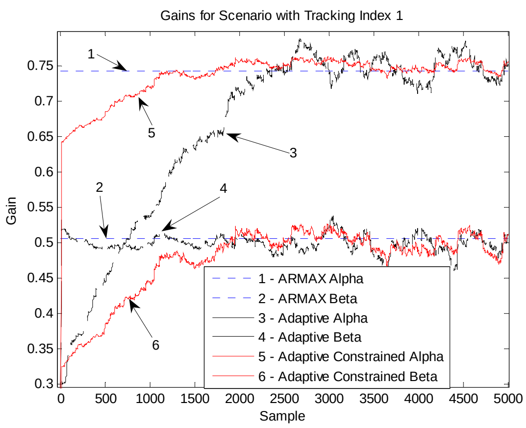

Fig. 5. Tracker gains: 1,2 – Method 1 (ARMAX), 3,4 – Method 2 (Mult.-Par.), 5,6 – Method 2 (Single-Par.).

Figs. 6-8 compare adaptive gains versus optimal solutions found analytically (for orders 2-3) or numerically (order 4). In these trials, the signal y_i is processed by several ways. First, by a multiple-parameter adaptive tracker constructed in accordance to the Method 1, then by a tracker due to the Method 2 and finally by a single-parameter variant (full or truncated) of the Method 2. As an applied filter converges and the innovation approaches the white-noise sequence, the kinematic gain is averaged over the remaining time and the resulting statistics, mean and standard deviation, are collected. This procedure is repeated 10 times and the final gain is averaged over 10 runs. These trials are performed with different Λ changing from 10^{-3} to 10^{3} (with Q=1 and R varied due to Λ).

Fig. 6 compares the adaptive α and β (obtained by various algorithms) versus the theoretical α (solid curve) and β (dash), respectively. The model (47) with N=2 provides a signal with the 1st-order trend. First, Fig. 6 depicts the adaptive α (diamond) and β (square) values outputted by the two-parameter adaptive filter. Both Method 1 and 2 gave like results. Next, Fig. 6 shows the adaptive α (plus) associated with the single-parameter tracker and, equivalently, its truncated version (in the plot two indistinguishable). In average, the adaptive α and β are shown to be close to analytically found optimal values.

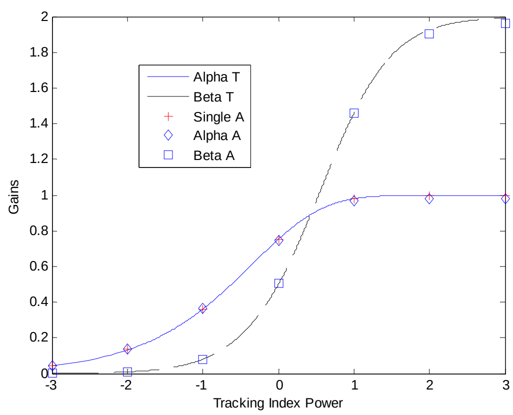

Fig. 6. Gains: 'T'-theory, 'A' - adaptive, 'Single' - single-parameter filter.

Similarly, the model (47) with N=3 was used to generate the signal with the 2nd-order trend. Fig. 7 illustrates performance of the 3-state adaptive tracking filter and its constrained variants, single-parameter and truncated. The adaptive α, β and γ obtained by different algorithms are compared to the theoretical α (solid curve) and β (dash) and γ (dotted), respectively. Fig. 7 plots the adaptive α (diamond), β (square) and γ (triangle) related to the three-parameter tracker implemented by the Method 1 or 2. In addition, Fig. 7 shows the adaptive α (plus) associated with the single-parameter tracker and, equivalently, with its truncated version (both indistinguishable). In average, the adaptive α, β and γ are close to the optimal solutions. As observed from all trials, the single-parameter version of the adaptive filter shows a lower gain-adjustment noise than its multiple-parameter counterpart. Monitoring is required only for the single-parameter filter utilizing the constraint (10) to keep $\alpha < 1$.

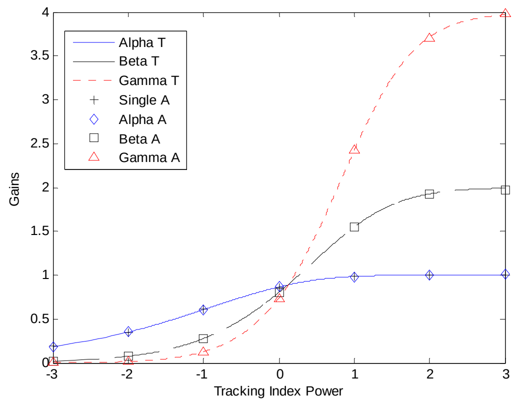

Fig. 7. Gains: 'T'-theory, 'A' - adaptive, 'Single' - single-parameter filter.

The truncated form doesn't deviate visibly from the full-gradient (single-parameter) adaptive filter.

Next, Fig. 8 exposes simulation results for a higher order model, namely, α-β-γ-λ filter. The experiments obey the previous scenario however with a signal generated due to the model (48) driven by the instantaneous random jerk-like noise. As a gold reference, we use a standard Kalman filter with known Q and R. As seen from Fig. 8, the adaptive gain closely fits 'optimal' values provided by the steady-state Kalman filter.

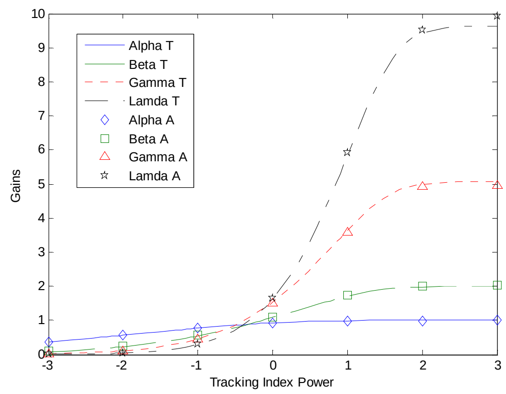

Fig. 8. Gains: 'T'-theory, 'A' - adaptive.

Apparently, a constrained variant of the 4-state filter is unavailable.

VII. Concluding Remarks

The steady-gain α-β, α-β-γ and α–β-γ-λ tracking filters are given a simple adaptive form when the filter gain is treated as an adaptation parameter. Importantly, for the kinematic model, the innovation-minimum-variance criterion results in the optimal Kalman gain. Accordingly, the innovation-based adaptive kinematic filter is asymptotically equivalent to the optimal steady-state Kalman filter. The resulting adaptive-gain kinematic filter may be implemented using two methods.

The Method 1 combines a stand-alone MA identification with the kinematic state estimator. Both stages are linked via the gain obtained from the MA parameters by a linear transform (20). The method performs, in series, difference of the observation signal, estimation of the MA parameters, their linear transformation into the kinematic gain-vector and, finally, the state estimation step.

An alternative Method 2 updates the gain directly using the innovation sequence produced by the Kalman filter.

The adaptive kinematic filter allows reasonable modifications. Thus, constraining components of the optimal gain translates the multiple-parameter adaptive kinematic filter into a beneficial single-parameter version.

The developed adaptive filters were tested using a wide range of conditions. In all runs, the gain approaches the optimal value predicted by theory. Asymptotically, both Methods 1 and 2 show similar results. The single-parameter version exhibits a lower gain-adjustment noise than the multiple-parameter filter.

Some other attractive properties of the adaptive kinematic filter deserve to be mentioned here. As shown [21], the adaptive gain successfully tracks rapidly varying maneuvering index Λ. The gain transient is analogous to that of the Kalman gain. To this end, adaptation translates the steady-state kinematic filter into a time-dependent version which by contrast to the time-varying Kalman counterpart does not require a priori data.

The suggested adaptive-gain filter should be compared with the covariance-matching and other related methods. Apparently the Method 1 should be beneficial to any other innivation based method because the kinematic gain is independent of the state errors. More thoroughul comparison of the suggested adaptive-gain filter with its numerous counterparts is postponed to future works.

Importantly, the simple kinematic model may be extended or viewed as a part of other models. In this context, one may consider the kinematic filter with exponentially-correlated states, with different types of dynamic input and measurement noise, with position-velocity or even position-velocity-acceleration measurements. Useful extended formulations arise also with the 2D and 3D tracking problems. Next, the kinematic model may be coupled with other commonly used models, e.g., with the sinusoidal or comb. There are also many nonlinear formulations resulting, e.g., from aggregation of the kinematic model with non-linear observation equations (bearing-only, etc). In all cases, the suggested approach may be adapted with certain efforts.

Finally, note that using an equivalent ARMA model behind the kinematic filter provides, in addition to the gain-adaptation property, a highly desired structure adaptability. Particularly, utilizing well-known procedures for the ARMA order-recursive estimation allows automatic identification of the kinematic filter order.

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