The R-Hyper-Panconnectedness of Faulty Crossed Cubes

Hon-Chan Chen*
Department of Information Management, National Chin-Yi University of Technology, Taiping, Taichung City 411, Taiwan.

* Corresponding author. Email: chenhc@ncut.edu.tw
Manuscript submitted November 12, 2018; accepted December 20, 2018.
doi: 10.17706/jcp.14.1.44-51

Abstract: Among the many kinds of network topologies, the crossed cube is one of the most popular. It is a variant of the hypercube with some attracting properties. A network topology is usually represented by a graph, where vertices and edges of the graph represent the nodes and communication links of the network. In this paper, we investigate the r-hyper-panconnectedness of faulty crossed cubes. A graph G is said to be r-hyper-panconnected if for any two distinct vertices x and y of G, it contains a Hamiltonian path P starting from x such that $d_P(x,y)=m$ for any integer m satisfying $r \le m \le |V(G)|-1$, in which $d_P(x,y)$ denotes the distance between x and y in P. Let CQ_n be an n-dimensional crossed cube. We demonstrate that for any one faulty vertex w of CQ_n and for any two distinct vertices x and y of $CQ_n-\{w\}$, $n \ge 5$, there exists a Hamiltonian path P of $CQ_n-\{w\}$ starting from x such that $d_P(x,y)=m$ for any integer m satisfying $2n \le m \le 2^n-2$. That is, the crossed cube of one vertex fault is $2n$-hyper-panconnected.

Key words: Crossed cube, hamiltonian path, path embedding, panconnectedness.

1. Introduction

In parallel and distributed computation, a network topology is essential and important because it determines the performance of a network. Among the many kinds of network topologies, the crossed cube is one of the most popular. It is a variant of the hypercube, and it has more attracting properties than the hypercube. A network topology is usually represented by a graph, where a vertex of the graph represents a node of the network whereas an edge of the graph represents a communication link of the network. Before introducing the crossed cube, we give some terminologies of graph theory.

Let $G=(V(G),E(G))$ be an undirected graph, where $V(G)$ and $E(G)$ are vertex set and edge set, respectively. Two vertices x and y of G are *adjacent* if $(x,y) \in E(G)$. A graph H is a *subgraph* of G, which is denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Moreover, if $V(H)=V(G)$, then H is a *spanning subgraph* of G, or we can say that H spans G. A *path* P of length k, $k \ge 1$, from vertex x to vertex y in G is a sequence of distinct vertices $[v_1, v_2, \cdots, v_{k+1}]$ such that $v_1=x$, $v_{k+1}=y$, and $(v_i, v_{i+1}) \in E(G)$ for $1 \le i \le k$. The *length* of path P, denoted by $l(P)$, is the number of edges of P. We can also write P as $[v_1, v_2, \cdots, v_i, Q, v_j, \cdots, v_{k+1}]$ for convenience if $Q=[v_i, \cdots, v_j]$ is a segment of P, where $i \le j$. The *reverse* of P, denoted by $rev(P)$, is defined as $rev(P)=[v_{k+1}, v_k, \cdots, v_1]$. The *distance* between two distinct vertices x and y in G, denoted by $d_G(x,y)$, is the length of the shortest path between x and y in G. A *cycle* is a closed path with at least three vertices, where the last vertex is adjacent to the first one.

A path is a *Hamiltonian path* of G if it spans G. Similarly, a cycle is a *Hamiltonian cycle* of G if it spans G. A graph G is *Hamiltonian* if it has a Hamiltonian cycle, and G is *Hamiltonian connected* if it has a Hamiltonian path between any two distinct vertices of G. Let F be a set of vertices of G. We use $G-F$ to denote the resulting graph after removing the vertices of F from G. Then, a Hamiltonian graph G is said to be *f-fault-tolerant Hamiltonian* if $G-F$ remains Hamiltonian for every $F \subseteq V(G) \cup E(G)$ with $|F| \leq f$. Similarly, a Hamiltonian connected graph G is said to be *f-fault-tolerant Hamiltonian connected* if $G-F$ remains Hamiltonian connected for every $F \subseteq V(G) \cup E(G)$ with $|F| \leq f$.

In general, paths of various lengths can support the flexibility of message transmission. A graph G is *panconnected* [1] if for any two distinct vertices x and y of G, it has a path of length m joining x and y for any integer m satisfying $d_G(x,y) \leq m \leq |V(G)|-1$. Furthermore, a graph G is said to be *hyper-panconnected* [2] if for any two distinct vertices x and y of G, it contains a Hamiltonian path P starting from x such that $d_P(x,y)=m$ for any integer m satisfying $d_G(x,y) \leq m \leq |V(G)|-1$. With the above definition, we can propose a loosed version of hyper-panconnectedness as follows: A graph G is said to be *r-hyper-panconnected* if for any two distinct vertices x and y of G, it has a Hamiltonian path P starting from x such that $d_P(x,y)=m$ for any integer m satisfying $r \leq m \leq |V(G)|-1$.

In this paper, we discuss the r-hyper-panconnectedness with respect to crossed cubes of faulty vertices. The formal definition of crossed cubes is introduced in Section 2. The hyper-panconnectedness of crossed cubes was first proposed in [3]. It is shown in [3] that for any two distinct vertices x and y of an n-dimensional crossed cube CQ_n, there exists a Hamiltonian path P of CQ_n starting from x such that $d_P(x,y)=m$ for any integer m satisfying $d_{CQ_n}(x,y) \leq m \leq 2^n-1$ and $m \neq d_{CQ_n}(x,y)+1$. This result is for a crossed cube without any faulty vertex. However, in practice, vertex faults and edge faults may happen when a network works. Therefore, it is meaningful to consider graph properties with respect to faulty networks. For simplicity as our first milestone, we consider that there exists one faulty vertex in the crossed cube. Although the number of faulty vertices is one, the result is not trivial. In this paper, we demonstrate that for any one faulty vertex w of CQ_n and for any two distinct vertices x and y of $CQ_n-\{w\}$, $n \geq 5$, there exists a Hamiltonian path P of $CQ_n-\{w\}$ starting from x such that $d_P(x,y)=m$ for any integer m satisfying $2n \leq m \leq 2^n-2$.

The remainder of this paper is organized as follows. In Section 2, the definition and some properties of crossed cubes are introduced. In Section 3, we propose and prove our main result of the r-hyper-panconnectedness of crossed cubes with one vertex fault. Finally, our concluding remarks are provided in Section 4.

2. Preliminaries

A crossed cube of n-dimensions, denoted by CQ_n, has 2^n vertices, and each vertex of CQ_n is identified by a unique n-bit binary string. Let $x=x_2x_1$ and $y=y_2y_1$ be two binary strings of length two. Then, x and y are *pair-related*, denoted by $x \sim y$, if and only if $(x,y) \in \{(00,00),(10,10),(01,11),(11,01)\}$. With the concept of "pair-related", we can define crossed cubes as follows [4]: The crossed cube is constructed recursively from CQ_1, in which CQ_1 is a complete graph with two vertices 0 and 1. For $n \geq 2$, CQ_n consists of two identical $(n-1)$ -dimensional crossed cubes CQ_{n-1}^0 and CQ_{n-1}^1, and a vertex $u=0u_{n-1}\cdots u_2u_1 \in V(CQ_{n-1}^0)$ is adjacent to a vertex $v=1v_{n-1}\cdots v_2v_1 \in V(CQ_{n-1}^1)$ in CQ_n if and only if (i) $u_{n-1}=v_{n-1}$ if n is even, and (ii) $u_{2i}u_{2i-1} \sim v_{2i}v_{2i-1}$ for all i, $1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Fig. 1 illustrates a CQ_3 and CQ_4.

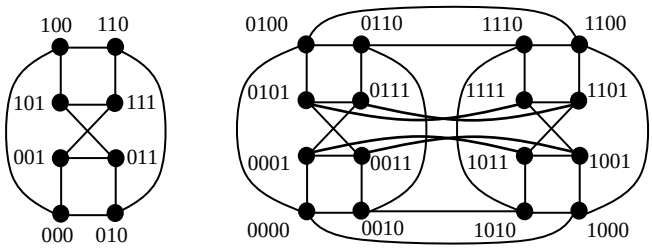

Fig. 1. CQ_3 and CQ_4.

Let $x = x_n x_{n-1} \cdots x_1$ and $y = y_n y_{n-1} \cdots y_1$ be any two vertices of CQ_n. Then, x is the *i-neighbor*, $1 \le i \le n$, of y, denoted by $x = (y)^i$, if the following four conditions are all satisfied: (i) $x_i \ne y_i$, (ii) $x_j = y_j$ for all j, $i+1 \le j \le n$, (iii) $x_{2k}x_{2k-1} \sim y_{2k}y_{2k-1}$ for all k, $1 \le k \le \left\lfloor \frac{i-1}{2} \right\rfloor$, and (iv) $x_{i-1} = y_{i-1}$ if i is even. The following are some properties that are useful to our main result.

Lemma 1. [5] *For any integer n, $n \ge 3$, CQ_n is (n–2)-fault-tolerant Hamiltonian and (n–3)-fault-tolerant Hamiltonian-connected.*

Lemma 2. [6] *Let w be any vertex of CQ_n and (x, y) be any edge of $CQ_n - \{w\}$, $n \ge 5$. Then, $CQ_n - \{w, x, y\}$ is Hamiltonian-connected.*

Lemma 3. [7] *Let u, v, x, and y be any four vertices of CQ_n, $n \ge 5$. Then, there exist two vertex-disjoint paths P_1 and P_2 in CQ_n such that (i) P_1 joins u and v with $l(P_1) = 2^{n-1} - 1$, and (ii) P_2 joins x and y with $l(P_2) = 2^{n-1} - 1$.*

Lemma 4. [3] *Let x and y be any two vertices of CQ_n, $n \ge 4$. For any integer m satisfying $d_{CQ_n}(x, y) \le m \le 2^n - 1$ and $m \ne d_{CQ_n}(x, y) + 1$, there exists a Hamiltonian path P of CQ_n starting from x with $d_P(x, y) = m$.*

3. The *R*-Hyper-Panconnectedness

In this section, we propose and prove the following theorem as our main result.

Theorem 1. *Let w be a vertex of CQ_n, $n \ge 5$. Moreover, let x and y be any two vertices of $CQ_n - \{w\}$. Then, there exists a Hamiltonian path P of $CQ_n - \{w\}$ starting from x such that $d_P(x, y) = m$ for any integer m satisfying $2n \le m \le 2^n - 2$.*

Proof. We prove this theorem by induction. The correctness of the induction base CQ_5 is verified by our computer program. As the inductive hypothesis, assume that this theorem holds for any CQ_k, $5 \le k \le n-1$. We show that this theorem also holds for CQ_n. Without loss of generality, we assume w is in CQ_{n-1}^0. According to the possibilities of x and y, we need to consider the following six cases.

Case 1. $\{x, y\} \subset V(CQ_{n-1}^0)$. In this case, we also need to distinguish four subcases.

Subcase 1.1. $2n \le m \le 2^{n-1} - 2$. By the inductive hypothesis, there exists a Hamiltonian path R of $CQ_{n-1}^0 - \{w\}$ starting from x such that $d_R(x, y) = m$. We can write R as $[x, R_1, y, R_2, p]$ for some vertex p. Note that $y = p$ (or $l(R_2) = 0$) if $d_R(x, y) = 2^{n-1} - 2$. Let z be a vertex of $CQ_{n-1}^1 - \{(p)^n\}$. By Lemma 1, there exists a Hamiltonian path S of CQ_{n-1}^1 joining $(p)^n$ and z. We set $P = [x, R_1, y, R_2, p, (p)^n, S, z]$, and P is a Hamiltonian path of $CQ_n - \{w\}$ such that $d_P(x, y) = m$.

Subcase 1.2. $m = 2^{n-1} - 1$. Let (p, q) be an edge of CQ_{n-1}^1 such that $\{(p)^n, (q)^n\} \cap \{w, x, y\} = \emptyset$. By Lemma 1, there exists a Hamiltonian path R of $CQ_{n-1}^0 - \{w, x\}$ joining y and $(p)^n$. Also by Lemma 1, there exists a Hamiltonian path S of $CQ_{n-1}^1 - \{p, q\}$ joining $(x)^n$ and $(y)^n$ with $l(S) = 2^{n-1} - 3$. We set $P = [x, (x)^n, S, (y)^n, y, R, (p)^n, p, q]$, then P is a Hamiltonian path of $CQ_n - \{w\}$ such that $d_P(x, y) = l(S) + 2 = m$.

Subcase 1.3. $m = 2^{n-1}$. Let p be a vertex of $CQ_{n-1}^0 - \{w, x, y\}$. By Lemma 1, we have a Hamiltonian path R of $CQ_{n-1}^0 - \{w, x\}$ joining y and p, and we have a Hamiltonian path S of $CQ_{n-1}^1 - \{(p)^n\}$ joining $(x)^n$ and $(y)^n$ with $l(S) = 2^{n-1} - 2$. Then, $P = [x, (x)^n, S, (y)^n, y, R, p, (p)^n]$ is a Hamiltonian path of $CQ_n - \{w\}$ such that $d_P(x, y) = l(S) + 2 = m$.

Subcase 1.4. $2^{n-1} + 1 \le m \le 2^n - 2$. By Lemma 1, there exists a Hamiltonian path R of $CQ_{n-1}^0 - \{w\}$ joining x and y. We can write R as $[x, R_1, p, q, R_2, y]$ for some adjacent vertices p and q, where $l(R_1) = m - 2^{n-1} - 1$. Note that $y = q$ if $m = 2^n - 2$. In CQ_{n-1}^1, there exists a Hamiltonian path S joining $(p)^n$ and $(q)^n$ with $l(S) = 2^{n-1} - 1$. We set $P = [x, R_1, p, (p)^n, S, (y)^n, y, R_2, q]$. Then, P is a Hamiltonian path of $CQ_n - \{w\}$, and $d_P(x, y) = l(R_1) + l(S) + 2 = m$.

Case 2. $\{x, y\} \subset V(CQ_{n-1}^1)$. We distinguish the following subcases.

Subcase 2.1. $2n \le m \le 2^{n-1} - 3$. By Lemma 4, there exists a Hamiltonian path S of CQ_{n-1}^1 starting from x such that $d_S(x, y) = m$. We can write S as $[x, S_1, y, S_2, p]$ for some vertex p, where $l(S_1) = m$. There are two conditions that should be considered.

Condition 2.1.1. $(p)^n \ne w$. Let z be a vertex of $CQ_{n-1}^0 - \{w, (p)^n\}$. Lemma 1 ensures that there exists a Hamiltonian path R of $CQ_{n-1}^0 - \{w\}$ joining $(p)^n$ and z. Then, $P = [x, S_1, y, S_2, p, (p)^n, R, z]$ is a Hamiltonian path of $CQ_n - \{w\}$ such that $d_P(x, y) = m$.

Condition 2.1.2. $(p)^n = w$. In this condition, we can rewrite S as $[x, S_1, y, S_2^*, s, t, p]$. By Lemma 1, there is a Hamiltonian path R of $CQ_{n-1}^0 - \{w\}$ joining $(s)^n$ and $(t)^n$. Then, $P = [x, S_1, y, S_2^*, s, (s)^n, R, (t)^n, t, p]$ is our required Hamiltonian path.

Subcase 2.2. $m = 2^{n-1} - 2$. Let p be a neighbor of y in CQ_{n-1}^1 with $p \notin \{x, (w)^n\}$. Besides, let z be a vertex of $CQ_{n-1}^0 - \{w, (p)^n\}$. By Lemma 1, there exists a Hamiltonian path S of $CQ_{n-1}^1 - \{p\}$ joining x and y with $l(S) = 2^{n-1} - 2$. Also by Lemma 1, there exists a Hamiltonian path R of $CQ_{n-1}^0 - \{w\}$ joining $(p)^n$ and z. We set $P = [x, S, y, p, (p)^n, R, z]$, and P is a Hamiltonian path of $CQ_n - \{w\}$ such that $d_P(x, y) = l(S) = m$.

Subcase 2.3. $m = 2^{n-1} - 1$. Distinguish the following two conditions.

Condition 2.3.1. $(y)^n \ne w$. Let z be a vertex of $CQ_{n-1}^0 - \{w, (y)^n\}$. By Lemma 1, we have a Hamiltonian path S of CQ_{n-1}^1 joining x and y with $l(S) = 2^{n-1} - 1$, and we have a Hamiltonian path R of $CQ_{n-1}^0 - \{w\}$ joining $(y)^n$ and z. Then, $P = [x, S, y, (y)^n, R, z]$ is a Hamiltonian path of $CQ_n - \{w\}$ such that $d_P(x, y) = l(S) = m$.

Condition 2.3.2. $(y)^n = w$. Let v be a neighbor of y in $CQ_{n-1}^1 - \{x\}$. Moreover, let (p, q) be an edge of CQ_{n-1}^0 with $\{p, q\} \cap \{w, (x)^n, (v)^n\} = \emptyset$. Lemma 2 ensures that there exists a Hamiltonian path R of $CQ_{n-1}^0 - \{w, p, q\}$ joining $(x)^n$ and $(v)^n$ with $l(R) = 2^{n-1} - 4$. By Lemma 1, there exists a Hamiltonian path S of $CQ_{n-1}^1 - \{x, v\}$ joining y and $(p)^n$. We set $P = [x, (x)^n, R, (v)^n, v, y, S, (p)^n, p, q]$. Because $d_P(x, y) = l(R) + 3 = m$, P is our required Hamiltonian path.

Subcase 2.4. $m = 2^{n-1}$. Let p be a neighbor of x and q be a neighbor of y in CQ_{n-1}^1 with $\{p, q\} \cap \{x, y, (w)^n\} = \emptyset$. Moreover, let (s, t) be an edge of $CQ_{n-1}^0 - \{w, (p)^n, (q)^n\}$ with $(s)^n \notin \{x, y\}$. By Lemma 2, we have a Hamiltonian path R of $CQ_{n-1}^0 - \{w, s, t\}$ joining $(p)^n$ and $(q)^n$ with $l(R) = 2^{n-1} - 4$. Also by Lemma 2, we have a Hamiltonian path S of $CQ_{n-1}^1 - \{x, p, q\}$ joining y and $(s)^n$. Then, $P = [x, p, (p)^n, R, (q)^n, q, y, S, (s)^n, s, t]$ is a Hamiltonian path of $CQ_n - \{w\}$ such that $d_P(x, y) = l(R) + 4 = m$.

Subcase 2.5. $m = 2^{n-1} + 1$. Let p be a neighbor of x and q be a neighbor of y in CQ_{n-1}^1 with $\{p, q\} \cap \{x, y, (w)^n\} = \emptyset$. Moreover, let z be a vertex of $CQ_{n-1}^0 - \{w, (p)^n, (q)^n\}$ with $(z)^n \notin \{x, y\}$. By Lemma 1, there exists a Hamiltonian path R of $CQ_{n-1}^0 - \{w, z\}$ joining $(p)^n$ and $(q)^n$ with $l(R) =$

$2^{n-1}-3$. According to Lemma 2, there exists a Hamiltonian path S of $CQ_{n-1}^1-\{x,p,q\}$ joining y and $(z)^n$. We set $P=[x,p,(p)^n,R,(q)^n,q,y,S,(z)^n,z]$, and $d_P(x,y)=l(R)+4=m$. Then, P is our required Hamiltonian path.

Subcase 2.6. $2^{n-1}+2\le m\le 2^n-3$. Let p be a neighbor of x in CQ_{n-1}^1 with $p\notin\{y,(w)^n\}$. By Lemma 1, there exists a Hamiltonian cycle C of $CQ_{n-1}^1-\{x,p\}$. We can write C as $[s,C_1,y,C_2,t,s]$, where $d_C(s,y)=m-2^{n-1}-1$. We need to consider the following two conditions.

Condition 2.6.1. $(s)^n\neq w$. By Lemma 1, there exists a Hamiltonian path R of $CQ_{n-1}^0-\{w\}$ joining $(p)^n$ and $(s)^n$ with $l(R)=2^{n-1}-2$. We set $P=[x,p,(p)^n,R,(s)^n,s,C_1,y,C_2,t]$, then $d_P(x,y)=l(R)+l(C_1)+3=(2^{n-1}-2)+(m-2^{n-1}-1)+3=m$. Thus, P is our required Hamiltonian path.

Condition 2.6.2. $(s)^n=w$. In this condition, we can rewrite C as $[t,s,C_1,y,C_2^*,u,t]$. By Lemma 1, there exists a Hamiltonian path R of $CQ_{n-1}^0-\{w,(u)^n\}$ joining $(p)^n$ and $(t)^n$ with $l(R)=2^{n-1}-3$. Then, $P=[x,p,(p)^n,R,(t)^n,t,s,C_1,y,C_2^*,u,(u)^n]$ is a Hamiltonian path of $CQ_n-\{w\}$. Because $d_P(x,y)=l(R)+l(C_1)+4=(2^{n-1}-3)+(m-2^{n-1}-1)+4=m$, P is our required Hamiltonian path.

Subcase 2.7. $m=2^n-2$. Let p be a neighbor of x in CQ_{n-1}^1 with $p\notin\{y,(w)^n\}$. Moreover, let u be a vertex of $CQ_{n-1}^1-\{x,y,p,(w)^n\}$. By Lemma 1, there exists a Hamiltonian path R of $CQ_{n-1}^0-\{w\}$ joining $(p)^n$ and $(u)^n$ with $l(R)=2^{n-1}-2$, and there exists a Hamiltonian path S of $CQ_{n-1}^1-\{x,p\}$ joining u and y with $l(S)=2^{n-1}-3$. Then, $P=[x,p,(p)^n,R,(u)^n,u,S,y]$ is a Hamiltonian path of $CQ_n-\{w\}$ such that $d_P(x,y)=l(R)+l(S)+3=(2^{n-1}-2)+(2^{n-1}-3)+3=m$.

Case 3. $x\in V(CQ_{n-1}^0)$, $y\in V(CQ_{n-1}^1)$, and $(x,y)\notin E(CQ_n)$. Distinguish the following subcases.

Subcase 3.1. $2n\le m\le 2^{n-1}-2$. By Lemma 4, there exists a Hamiltonian path S of CQ_{n-1}^1 starting from $(x)^n$ such that $d_S((x)^n,y)=m-1$. We can write S as $[(x)^n,S_1,y,S_2,p]$ for some vertex p, where $l(S_1)=m-1$. We need to consider two conditions.

Condition 3.1.1. $(p)^n\neq w$. Let z be a vertex of $CQ_{n-1}^0-\{w,x\}$. By Lemma 1, we have a Hamiltonian path R of $CQ_{n-1}^0-\{w,x\}$ joining $(p)^n$ and z. Then, $P=[x,(x)^n,S_1,y,S_2,p,(p)^n,R,z]$ is a Hamiltonian path of $CQ_n-\{w\}$ such that $d_P(x,y)=l(S_1)+1=m$.

Condition 3.1.2. $(p)^n=w$. In this condition, we can rewrite S as $[(x)^n,S_1,y,S_2^*,s,t,p]$. By Lemma 1, there is a Hamiltonian path R of $CQ_{n-1}^0-\{w,x\}$ joining $(s)^n$ and $(t)^n$. Then, $P=[x,(x)^n,S_1,y,S_2^*,s,(s)^n,R,(t)^n,t,p]$ is our required Hamiltonian path because $d_P(x,y)=l(S_1)+1=m$.

Subcase 3.2. $m=2^{n-1}-1$. Let p be a neighbor of y in CQ_{n-1}^1 with $(p)^n\neq w$. Moreover, let z be a vertex of $CQ_{n-1}^0-\{w,x,(p)^n,(y)^n\}$. By Lemma 1, there exists a Hamiltonian path R of $CQ_{n-1}^0-\{w,z\}$ joining x and $(p)^n$ with $l(R)=2^{n-1}-3$, and there exists a Hamiltonian path S of $CQ_{n-1}^1-\{p\}$ joining y and $(z)^n$. We set $P=[x,R,(p)^n,p,y,S,(z)^n,z]$. Then, P is a Hamiltonian path of $CQ_n-\{w\}$ such that $d_P(x,y)=l(R)+2=m$.

Subcase 3.3. $m=2^{n-1}$. Let p be a neighbor of y in CQ_{n-1}^1 with $(p)^n\notin\{w,x\}$. Moreover, let z be a vertex of $CQ_{n-1}^1-\{y,p\}$. Lemma 1 ensures that we have a Hamiltonian path R of $CQ_{n-1}^0-\{w\}$ joining x and $(p)^n$ with $l(R)=2^{n-1}-2$, and we have a Hamiltonian path S of $CQ_{n-1}^1-\{p\}$ joining y and z. Then, $P=[x,R,(p)^n,p,y,S,z]$ is our required Hamiltonian path because $d_P(x,y)=l(R)+2=m$.

Subcase 3.4. $m=2^{n-1}+1$. Let p be a neighbor of y in CQ_{n-1}^1 and (p,q) be an edge of CQ_{n-1}^1 with $(q)^n\notin\{w,x\}$. Moreover, let z be a vertex of $CQ_{n-1}^1-\{y,p,q\}$. By Lemma 1, there is a Hamiltonian path R of $CQ_{n-1}^0-\{w\}$ joining x and $(q)^n$ with $l(R)=2^{n-1}-2$, and there exists a Hamiltonian path S of $CQ_{n-1}^1-\{p,q\}$ joining y and z. We set $P=[x,R,(q)^n,q,p,y,S,z]$, and P is a Hamiltonian path of $CQ_n-\{w\}$ such that $d_P(x,y)=l(R)+3=m$.

Subcase 3.5. $2^{n-1}+2 \le m \le 2^n-2$. Let p be a neighbor of y in CQ_{n-1}^1 with $(p)^n \notin \{w,x\}$. By Lemma 1, we have a Hamiltonian path R of $CQ_{n-1}^0-\{w\}$ with $l(R)=2^{n-1}-2$. By Lemma 4, we have a Hamiltonian path S of CQ_{n-1}^1 starting from p such that $d_S(p,y)=m-2^{n-1}+1$. We can write S as $[p,S_1,y,S_2,z]$ for some vertex z, where $l(S_1)=m-2^{n-1}+1$. Note that $y=z$ if $l(S_2)=0$. Then, $P=[x,R,(p)^n,p,S_1,y,S_2,z]$ is a Hamiltonian path of $CQ_n-\{w\}$ such that $d_P(x,y)=l(R)+l(S_1)+1=(2^{n-1}-2)+(m-2^{n-1}+1)+1=m$.

Case 4. $x \in V(CQ_{n-1}^0)$, $y \in V(CQ_{n-1}^1)$, and $(x,y) \in E(CQ_n)$. Consider the following subcases.

Subcase 4.1. $2n \le m \le 2^{n-1}-1$. Let p be a neighbor of y in CQ_{n-1}^1 with $(p)^n \ne w$. By the inductive hypothesis, there exists a Hamiltonian path R of $CQ_{n-1}^0-\{w\}$ starting from $(p)^n$ such that $d_R((p)^n,x)=m-2$. We can write R as $[(p)^n,R_1,x,q,R_2,s]$ for some vertex s, where $l(R_1)=m-2$. By Lemma 1, there exists a Hamiltonian path S of $CQ_{n-1}^1-\{p\}$ joining y and $(s)^n$. We set $P=[x,rev(R_1),(p)^n,p,y,S,(s)^n,s,rev(R_2),q]$. Then, P is a Hamiltonian path of $CQ_n-\{w\}$ such that $d_P(x,y)=l(R_1)+2=m$.

Subcase 4.2. $2^{n-1} \le m \le 2^{n-1}+2^{n-2}-4$. Let p be a vertex of $CQ_{n-1}^0-\{w,x\}$. By the inductive hypothesis, there exists a Hamiltonian path R of $CQ_{n-1}^0-\{w\}$ starting from x such that $d_R(x,p)=m-2^{n-2}$. We can write R as $[x,R_1,p,q,R_2,s]$ for some vertex s, where $l(R_1)=m-2^{n-2}$. Let t be a neighbor of y in $CQ_{n-1}^1-\{(p)^n,(s)^n\}$. By Lemma 3, there exist two paths S_1 and S_2 in CQ_{n-1}^1 such that (i) S_1 joins y and $(p)^n$ with $l(S_1)=2^{n-2}-1$, and (ii) S_2 joins t and $(s)^n$ with $l(S_2)=2^{n-2}-1$. Then, $P=[x,R_1,p,(p)^n,S_1,y,t,S_2,(s)^n,s,rev(R_2),q]$ is a Hamiltonian path of $CQ_n-\{w\}$ such that $d_P(x,y)=l(R_1)+l(S_1)+1=(m-2^{n-2})+(2^{n-2}-1)+1=m$.

Subcase 4.3. $2^{n-1}+2^{n-2}-3 \le m \le 2^n-2$. Let p be a vertex of $CQ_{n-1}^0-\{w,x\}$. By Lemma 1, there is a Hamiltonian path R of $CQ_{n-1}^0-\{w\}$ joining x and p with $l(R)=2^{n-1}-2$. In CQ_{n-1}^1, by Lemma 4, there exists a Hamiltonian path S starting from $(p)^n$ such that $d_S((p)^n,y)=m-2^{n-1}+1$. We can write S as $[(p)^n,S_1,y,S_2,z]$ for some vertex z, where $l(S_1)=d_S((p)^n,y)$. Then, $P=[x,R,p,(p)^n,S_1,y,S_2,z]$ is a Hamiltonian path of $CQ_n-\{w\}$. Because $d_P(x,y)=l(R)+l(S_1)+1=(2^{n-1}-2)+(m-2^{n-1}+1)+1=m$, P is our required Hamiltonian path.

Case 5. $x \in V(CQ_{n-1}^1)$, $y \in V(CQ_{n-1}^0)$, and $(x,y) \notin E(CQ_n)$. There are five subcases that should be considered.

Subcase 5.1. $2n \le m \le 2^{n-1}-1$. Let p be a neighbor of x in CQ_{n-1}^1 with $(p)^n \ne w$ By the inductive hypothesis, there exists a Hamiltonian path R of $CQ_{n-1}^0-\{w\}$ starting from $(p)^n$ such that $d_R((p)^n,y)=m-2$. We can write R as $[(p)^n,R_1,y,R_2,q]$ for some vertex q, where $l(R_1)=m-2$. Let z be a vertex of $CQ_{n-1}^1-\{x,p,(q)^n\}$. By Lemma 1, there exists a Hamiltonian path S of $CQ_{n-1}^1-\{x,p\}$ joining $(q)^n$ and z. We set $P=[x,p,(p)^n,R_1,y,R_2,q,(q)^n,S,z]$. Then, P is a Hamiltonian path of $CQ_n-\{w\}$ such that $d_P(x,y)=l(R_1)+2=m$

Subcase 5.2. $m=2^{n-1}$. Let z be a vertex of $CQ_{n-1}^0-\{w,y\}$. By Lemma 1, we have a Hamiltonian path S of CQ_{n-1}^1 joining x and $(y)^n$ with $l(S)=2^{n-1}-1$. Also by Lemma 1, we have a Hamiltonian path R of $CQ_{n-1}^0-\{w\}$ joining y and z. Then, $P=[x,S,(y)^n,y,R,z]$ is our required Hamiltonian path because $d_P(x,y)=l(S)+1=m$.

Subcase 5.3. $2^{n-1}+1 \le m \le 2^n-4$. By Lemma 1, there is a Hamiltonian cycle C of $CQ_{n-1}^0-\{w\}$. We can write C as $[y,C_1,p,s,C_2,y]$, where $l(C_1)=m-2^{n-1}$. We distinguish the following two conditions.

Condition 5.3.1. $(p)^n \ne x$. According to Lemma 1, we have a Hamiltonian path S of CQ_{n-1}^1 joining x and $(p)^n$ with $l(S)=2^{n-1}-1$. Then, $P=[x,S,(p)^n,p,rev(C_1),y,rev(C_2),s]$ and $d_P(x,y)=l(S)+l(C_1)+1=(2^{n-1}-1)+(m-2^{n-1})+1=m$. Thus, P is our required Hamiltonian path.

Condition 5.3.2. $(p)^n = x$. In this condition, we can rewrite C as $[y, C_1, p, s, t, C_2^*, y]$. By Lemma 1, there is a Hamiltonian path S of $CQ_{n-1}^1 - \{(t)^n\}$ joining x and $(s)^n$ with $l(S) = 2^{n-1} - 2$. We set $P = [x, S, (s)^n, s, p, rev(C_1), y, rev(C_2^*), t, (t)^n]$ and $d_P(x, y) = l(S) + l(C_1) + 2 = (2^{n-1} - 2) + (m - 2^{n-1}) + 2 = m$. Then, P is our required Hamiltonian path.

Subcase 5.4. $m = 2^n - 3$. Let p be a vertex of $CQ_{n-1}^1 - \{x\}$ with $(p)^n \neq w$. Besides, let z be a neighbor of y in $CQ_{n-1}^0 - \{w, y, (p)^n\}$. Lemma 1 ensures that we have a Hamiltonian path S of CQ_{n-1}^1 joining x and p with $l(S) = 2^{n-1} - 1$, and we have a Hamiltonian path R of $CQ_{n-1}^0 - \{w, z\}$ joining $(p)^n$ and y with $l(R) = 2^{n-1} - 3$. By setting $P = [x, S, p, (p)^n, R, y, z]$, we obtain that $d_P(x, y) = l(S) + l(R) + 1 = (2^{n-1} - 1) + (2^{n-1} - 3) + 1 = m$ and P is our required Hamiltonian path.

Subcase 5.5. $m = 2^n - 2$. Let p be a vertex of $CQ_{n-1}^1 - \{x\}$ with $(p)^n \neq w$. By Lemma 1, there exists a Hamiltonian path S of CQ_{n-1}^1 joining x and p with $l(S) = 2^{n-1} - 1$, and there exists a Hamiltonian path R of $CQ_{n-1}^0 - \{w\}$ joining $(p)^n$ and y with $l(R) = 2^{n-1} - 2$. Then, $P = [x, S, p, (p)^n, R, y]$ is a Hamiltonian path of $CQ_n - \{w\}$ such that $d_P(x, y) = l(S) + l(R) + 1 = (2^{n-1} - 1) + (2^{n-1} - 2) + 1 = m$.

Case 6. $x \in V(CQ_{n-1}^1)$, $y \in V(CQ_{n-1}^0)$, and $(x, y) \in E(CQ_n)$. We distinguish the following subcases.

Subcase 6.1. $2n \leq m \leq 2^{n-1} - 1$. Let p be a neighbor of $CQ_{n-1}^1 - \{x\}$ with $(p)^n \neq w$. By the inductive hypothesis, we have a Hamiltonian path R of $CQ_{n-1}^0 - \{w\}$ starting from $(p)^n$ such that $d_R((p)^n, y) = m - 2$. We can write R as $[(p)^n, R_1, y, R_2, q]$ for some vertex q, in which $l(R_1) = m - 2$. Let z be a vertex of $CQ_{n-1}^1 - \{x, p, (q)^n\}$. By Lemma 1, there is a Hamiltonian path S of $CQ_{n-1}^1 - \{x, p\}$ joining $(q)^n$ and z. Then, $P = [x, p, (p)^n, R_1, y, R_2, q, (q)^n, S, z]$ is a Hamiltonian path of $CQ_n - \{w\}$ such that $d_P(x, y) = l(R_1) + 2 = m$.

Subcase 6.2. $m = 2^{n-1}$. Let p be a neighbor of y in $CQ_{n-1}^0 - \{w\}$. Moreover, let z be a vertex of $CQ_{n-1}^1 - \{x, (p)^n\}$. By Lemma 1, there is a Hamiltonian path S of $CQ_{n-1}^1 - \{z\}$ joining x and $(p)^n$ with $l(S) = 2^{n-1} - 2$. Also by Lemma 1, there is a Hamiltonian path R of $CQ_{n-1}^0 - \{w, p\}$ joining y and $(z)^n$. We set $P = [x, S, (p)^n, p, y, R, (z)^n, z]$ and $d_P(x, y) = l(S) + 2 = m$. Thus, P is our required Hamiltonian path.

Subcase 6.3. $2^{n-1} + 1 \leq m \leq 2^{n-1} + 2^{n-2} - 3$. Let p be a vertex of $CQ_{n-1}^1 - \{x\}$ with $(p)^n \neq w$. By the inductive hypothesis, there exists a Hamiltonian path R of $CQ_{n-1}^0 - \{w\}$ starting from $(p)^n$ such that $d_R((p)^n, y) = m - 2^{n-2}$. We can write R as $[(p)^n, R_1, y, R_2, q]$ for some vertex q, where $l(R_1) = m - 2^{n-2}$. Let z be a vertex of $CQ_{n-1}^1 - \{x, p, (q)^n\}$. By Lemma 3, there exist two paths S_1 and S_2 such that (i) S_1 joining x and p with $l(S_1) = 2^{n-2} - 1$, and (ii) S_2 joining $(q)^n$ and z with $l(S_2) = 2^{n-2} - 1$. Then, $P = [x, S_1, p, (p)^n, R_1, y, R_2, q, (q)^n, S_2, z]$ is a Hamiltonian path of $CQ_n - \{w\}$ such that $d_P(x, y) = l(S_1) + l(R_1) + 1 = (2^{n-2} - 1) + (m - 2^{n-2}) + 1 = m$, and P is our required Hamiltonian path.

Subcase 6.4. $2^{n-1} + 2^{n-2} - 2 \leq m \leq 2^n - 2$. Let p be a vertex of $CQ_{n-1}^0 - \{w, y\}$. Lemma 1 ensures that there exists a Hamiltonian path S of CQ_{n-1}^1 joining x and $(p)^n$ with $l(S) = 2^{n-1} - 1$. By the inductive hypothesis, there exists a Hamiltonian path R of $CQ_{n-1}^0 - \{w\}$ starting from p such that $d_R(p, y) = m - 2^{n-1}$. We can write R as $[p, R_1, y, R_2, z]$ for some vertex z, where $l(R_1) = m - 2^{n-1}$. We set $P = [x, S, (p)^n, p, R_1, y, R_2, z]$ and $d_P(x, y) = l(S) + l(R_1) + 1 = (2^{n-1} - 1) + (m - 2^{n-1}) + 1 = m$. Thus, P is our required Hamiltonian path.

The above argument completes the proof. □

4. Concluding Remarks

In parallel and distributed computing, paths are fundamental topologies. If paths of different lengths can be embedded, we can adjust the number of simulated processors and increase the flexibility of demand in

the network. In this paper, we investigate the r-hyper-panconnectedness of n-dimensional crossed cubes CQ_n. We demonstrate that for any one faulty vertex w of CQ_n, $n \geq 5$, and for any two distinct vertices x and y of $CQ_n - \{w\}$, there exists a Hamiltonian path P of $CQ_n - \{w\}$ starting from x such that $d_P(x, y) = m$ for any integer m satisfying $2n \leq m \leq 2^n - 2$. In other words, the crossed cube of one vertex fault is $2n$-hyper-panconnected. As our future work for the r-hyper-panconnectedness, we are interested in finding a tighter bound for the r value; i.e., $r < 2n$.

References

[1] Alavi, Y., & Williamson, J. E. (1975). Panconnected graphs. *Studia Scientiarum Mathematicarum Hungarica*, *10(1-2),* 19-22.

[2] Kung, T. L., Hsu, L. H., & Wu, J. J. (2011). Hyper-panconnectedness of the locally twisted cube. *Proceedings of the 28th Workshop on Combinatorial Mathematics and Computation Theory* (pp. 333-339). Taiwan.

[3] Chen, H. C., Kung, T. L., & Hsu, L. H. (2011). Embedding a hamiltonian cycle in the crossed cube with two required vertices in the fixed positions. *Applied Mathematics and Computation*, *217(24),* 10058-10065.

[4] Efe, K. (1991). A variation on the hypercube with lower diameter. *IEEE Transactions on Computers*, *40(11),* 1312-1316.

[5] Huang, W. T., Chuang, Y. C., Tan, J. J. M., & Hsu, L. H. (2002). On the fault-tolerant Hamiltonicity of faulty crossed cubes. *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, *E85–A(6),* 1359-1370.

[6] Chen, H. C., Kung, T. L., & Hsu, L. Y. (2015). 2-disjoint-path-coverable panconnectedness of crossed cubes. *Journal of Supercomputing*, *71(7),* 2767-2782.

[7] Lai, P. L., & Hsu, H. C. (2008). The two-equal-disjoint path cover problem of matching composition network. *Information Processing Letters*, *107(1),* 18-23.

Hon-Chan Chen received the BS, MS, and Ph.D degrees in information management from National Taiwan University of Science and Technology, Taipei, Taiwan, in 1992, 1994, and 1998. From 2000 to 2002, he was an assistant professor in National Kaohsiung Marine University, Kaohsiung, Taiwan. Since 2002, he has been teaching in the Department of Information Management, National Chin-Yi University of Technology, Taichung, Taiwan. He is currently an associate professor. His research interests include algorithms, parallel processing, and graph theory.